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A Probabilistic Approach to the Asymptotic Distribution of Sums of Independent, Identically Distributed Random Variables

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1. RESULTS

In this paper we present a new unified probabilistic approach to one of the most classical problems of probability theory, the problem of the asymptotic distribution of sums of independent and identically distributed random variables and the corresponding lightly trimmed sums formed when a fixed number of large and small summands are discarded from the full partial sum at each stage n. This unified approach is based upon the asymptotic behavior of the uniform empirical distribution function in conjunction with an integral representation of all these sums, in which representation the basic ingredients are the uniform empirical distribution function and the inverse or quantile function of the underlying distribution function. The only places where we rely upon Fourier analysis is the uniqueness of the components (Lévy measures and constants) of an infinitely divisible characteristic function and, indirectly, where we use Sato's

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estimates of the tail probabilities of infinitely divisible distributions in the proof of the first of our twelve corollaries. On the other hand, as both a consequence and an integral part of our approach, a representation for any infinitely divisible random variable is given.

First we formulate and prove general theorems describing the asymptotic distribution of arbitrary lightly trimmed (as a special case, untrimmed full) sums along subsequences of the positive integers (Theorems 1, 2, and 4) under conditions which are shown to be necessary in Theorem 5. The first two theorems also detail the fine asymptotic structure of our sums in the sense that they show which portions of the sums contribute the ingredients of the limiting infinitely divisible law or do not contribute anything at all and, taking into account Theorems 3 and 4, also demonstrate how the Lévy measures of these limiting laws arise.

In a sequence of corollaries (Corollaries 1–5, 7–12) to these general theorems we obtain necessary and sufficient conditions for these lightly trimmed or untrimmed sums to be in the domain of attraction or partial attraction of a normal or non-normal stable law, in the domain of partial attraction of some infinitely divisible law and a necessary and sufficient condition for the stochastic compactness and subsequential compactness of these sums. For further interesting consequences see Corollary 6 and the discussion in Section 4.

Naturally enough, since our new probabilistic approach to this classical field of probability theory is partly based on the behavior of the underlying quantile function as described above, the analytic conditions that our method yields are all expressed in terms of this function. These conditions are derived independently of the existing literature on conditions usually formulated by means of the underlying distribution function and should therefore be of independent interest. Connections with the literature (classical and recent) on this subject will be pointed out in a discussion placed in Section 4. All the proofs are in the second and third sections. The equivalence of one of our analytic conditions and the classical condition concerning stochastic compactness is shown in Section 5.

First we introduce some basic notation.

Let $X_1, X_2, \ldots$ be a sequence of independent random variables with a common (right-continuous) non-degenerate distribution function F, and for each integer $n \geq 1$, let $X_{1,n} \leq \cdots \leq X_{n,n}$ denote the order statistics based on the sample $X_1, \ldots, X_n$. Introduce the left-continuous inverse or quantile function Q of F defined as

$$Q(s) = \inf\{x\colon F(x) \geq s\}, \qquad 0 < s \leq 1, \qquad Q(0) = Q(0+),$$

and for $0 < s < 1 - t < 1$ consider the truncated variance function

$$\sigma^2(s, 1-t) = \int_s^{1-t}\int_s^{1-t} (u \wedge v - uv)\, dQ(u)\, dQ(v),$$

where $u \wedge v = \min(u, v)$ and $u \vee v = \max(u, v)$. For the special case $t = s$ we shall often use the abbreviation $\sigma^2(s) = \sigma^2(s, 1-s)$ throughout. Let m and k be fixed non-negative integers, and for each integer $n \geq 2(m \vee k + 1)$ set

$$a(n) = n^{1/2}\sigma(1/n)$$

and

$$\mu_{m,k}(n) = \int_{(m+1)/n}^{1-(k+1)/n} Q(u+)\,du,$$

where, for $0 \leq u < 1$, $Q(u+) = \lim_{v \downarrow u} Q(v)$.

Let $U_{1,n} \leq \dots \leq U_{n,n}$ be uniform $(0,1)$ order statistics. Then for each $n \geq 1$ we have the distributional equality

$$(X_{1,n}, \dots, X_{n,n}) =_D (Q(U_{1,n}), \dots, Q(U_{n,n})). \tag{1.1}$$

Let α_n be any sequence of positive constants such that

$$\alpha_n \downarrow 0 \quad \text{and } P\{F_n\} := P\{\alpha_n \leq U_{1,n} \leq U_{n,n} \leq 1 - \alpha_n\} \to 1 \qquad \text{as } n \to \infty. \tag{1.2}$$

(This holds if and only if $n\alpha_n \to 0$ as $n \to \infty$.) For each integer n large enough to make $a(n) > 0$ we define two right-continuous, non-decreasing functions on $(0, \infty)$ by putting

$$\psi_1(n, s) = \begin{cases} Q(s/n+)/a(n), & 0 < s \leq n - n\alpha_n, \\ Q((1-\alpha_n)+)/a(n), & n - n\alpha_n < s < \infty, \end{cases}$$

and

$$\psi_2(n, s) = \begin{cases} -Q(1 - s/n)/a(n), & 0 < s \leq n - n\alpha_n, \\ -Q(\alpha_n)/a(n), & n - n\alpha_n < s < \infty. \end{cases}$$

Let $N(\mu, \sigma^2)$ denote a normal random variable with mean μ and variance σ^2, which is understood to be the constant μ if $\sigma = 0$, and let $\to_D$ denote convergence in distribution and $\to_p$ denote convergence in probability. The first two theorems that follow contain our basic sufficiency results.

THEOREM 1. *Assume that there exists a subsequence $\{n_1\}$ of the positive integers such that for two (necessarily) non-decreasing, non-positive, right-continuous functions ψ_1 and ψ_2 defined on $(0, \infty)$ we have*

$$\psi_j(n_1, s) \to \psi_j(s) \qquad \textit{as } n_1 \to \infty \tag{1.3}$$

at every continuity point $s \in (0, \infty)$ *of* ψ_j, $j = 1, 2$.

(i) *If* $\psi_1 = \psi_2 \equiv 0$, *then for all fixed* $m \geq 0$ *and* $k \geq 0$, *as* $n_1 \to \infty$,

$$\frac{1}{a(n_1)}\left\{\sum_{j=m+1}^{n_1-k} X_{j,n_1} - n_1\mu_{m,k}(n_1)\right\} \to_D N(0,1). \tag{1.4}$$

(ii) *For arbitrary* ψ_1 *and* ψ_2 *there exist two sequences* $\{l_{n_1}\}$ *and* $\{r_{n_1}\}$ *of positive integers such that, as* $n_1 \to \infty$, $l_{n_1} \to \infty$,

$$r_{n_1}/n_1 \to 0, \tag{1.5}$$

$$l_{n_1}/r_{n_1} \to 0, \tag{1.6}$$

and for any pair of fixed $m \geq 0$ *and* $k \geq 0$,

$$\begin{aligned} &\frac{1}{a(n_1)}\left\{\sum_{j=l_{n_1}+1}^{r_{n_1}} X_{j,n_1} - n_1\int_{(l_{n_1}+1)/n_1}^{(r_{n_1}+1)/n_1} Q(u+)\,du\right\} \to_p 0,\\ &\frac{1}{a(n_1)}\left\{\sum_{j=n_1-r_{n_1}+1}^{n_1-l_{n_1}} X_{j,n_1} - n_1\int_{1-(r_{n_1}+1)/n_1}^{1-(l_{n_1}+1)/n_1} Q(u+)\,du\right\} \to_p 0, \end{aligned} \tag{1.7}$$

and

$$\begin{aligned} &\frac{1}{a(n_1)}\left\{\sum_{j=m+1}^{l_{n_1}} X_{j,n_1} - n_1\int_{(m+1)/n_1}^{(l_{n_1}+1)/n_1} Q(u+)\,du\right\} \to_D V_m^{(1)},\\ &\frac{1}{a(n_1)}\left\{\sum_{j=n_1-l_{n_1}+1}^{n_1-k} X_{j,n_1} - n_1\int_{1-(l_{n_1}+1)/n_1}^{1-(k+1)/n_1} Q(u+)\,du\right\} \to_D V_k^{(2)}, \end{aligned} \tag{1.8}$$

where, with independent left-continuous standard Poisson processes N_j *with jump-points* $S_1^{(j)}, S_2^{(j)}, \ldots$, $j = 1, 2$,

$$\begin{aligned} V_m^{(1)} = &\int_{S_{m+1}^{(1)}}^{\infty} (u - N_1(u))\,d\psi_1(u) + \int_1^{S_{m+1}^{(1)}} u\,d\psi_1(u) - m\psi_1\left(S_{m+1}^{(1)}\right)\\ &+ \int_1^{m+1}\psi_1(u)\,du + \psi_1(1) \end{aligned}$$

and

$$\begin{aligned} V_k^{(2)} = &-\int_{S_{k+1}^{(2)}}^{\infty} (u - N_2(u))\,d\psi_2(u) - \int_1^{S_{k+1}^{(2)}} u\,d\psi_2(u) + k\psi_2\left(S_{k+1}^{(2)}\right)\\ &- \int_1^{k+1}\psi_2(u)\,du - \psi_2(1), \end{aligned}$$

and where these limiting random variables $V_m^{(1)}$ *and* $V_k^{(2)}$ *are non-degenerate if* $\psi_1 \not\equiv 0$ *and* $\psi_2 \not\equiv 0$, *respectively. Furthermore, if* $\{n_2\}$ *is a subsequence of* $\{n_1\}$ *such that*

$$\sum(r_{n_2}, n_2) = \frac{\sigma\big((r_{n_2}+1)/n_2, 1-(r_{n_2}+1)/n_2\big)}{\sigma(1/n_2, 1-1/n_2)} \to \sigma \qquad \text{as } n_2 \to \infty,$$

where necessarily $0 \le \sigma \le 1$ *since* $0 \le \Sigma(r_{n_1}, n_1) \le 1$ (*and hence such an* $\{n_2\}$ *always exists*) *then*

$$\frac{1}{a(n_2)}\left\{\sum_{j=r_{n_2}+1}^{n_2-r_{n_2}} X_{j,n_2} - n_2\int_{(r_{n_2}+1)/n_2}^{1-(r_{n_2}+1)/n_2} Q(u+)\,du\right\} \to_D N(0,\sigma^2), \quad (1.9)$$

and the three limiting random variables $N(0,\sigma^2)$, $V_m^{(1)}$, *and* $V_k^{(2)}$ *are independent. Moreover, whenever* $\sigma > 0$, *we have*

$$\sigma\big((l_{n_2}+1)/n_2, 1-(l_{n_2}+1)/n_2\big)/\sigma\big((r_{n_2}+1)/n_2, 1-(r_{n_2}+1)/n_2\big) \to 1 \qquad \text{as } n_2 \to \infty. \quad (1.10)$$

The case $\sigma = 0$ of Theorem 1 indeed occurs: for example, as the proof of Corollary 3 will show, limiting stable distributions arise this way. The next theorem covers the case when a normalizing factor A_n diverging faster than $a(n)$ is needed. As a construction in the proof of Corollary 10 will show, this is not an empty case either. See the comment following the proof of Corollary 10.

THEOREM 2. *Assume that there exists a sequence* $\{n_1\}$ *of positive integers such that for a sequence of positive constants* A_{n_1} *and two nondecreasing, non-positive, right-continuous functions* ψ_1 *and* ψ_2 *we have*

$$\frac{a(n_1)}{A_{n_1}}\psi_j(n_1, s) \to \psi_j(s) \qquad \text{as } n_1 \to \infty \quad (1.11)$$

at every continuity point $s \in (0,\infty)$ *of* ψ_j, $j = 1, 2$, *and*

$$a(n_1)/A_{n_1} \to 0 \qquad \text{as } n_1 \to \infty. \quad (1.12)$$

Then there exists a sequence $\{l_{n_1}\}$ *of positive integers such that, as* $n_1 \to \infty$,

$$l_{n_1} \to \infty \qquad \text{and} \qquad l_{n_1}/n_1 \to 0, \quad (1.13)$$

$$\frac{1}{A_{n_1}}\left\{\sum_{j=l_{n_1}+1}^{n_1-l_{n_1}} X_{j,n_1} - n_1\int_{(l_{n_1}+1)/n_1}^{1-(l_{n_1}+1)/n_1} Q(u+)\,du\right\} \to_p 0, \quad (1.14)$$

and for any pair of fixed $m \geq 0$ and $k \geq 0$,

$$\frac{1}{A_{n_1}}\left\{\sum_{j=m+1}^{l_{n_1}} X_{j,n_1} - n_1\int_{(m+1)/n_1}^{(l_{n_1}+1)/n_1} Q(u+)\,du\right\} \to_D V_m^{(1)}$$

and (1.15)

$$\frac{1}{A_{n_1}}\left\{\sum_{j=n_1-l_{n_1}+1}^{n_1-k} X_{j,n_1} - n_1\int_{1-(l_{n_1}+1)/n_1}^{1-(k+1)/n_1} Q(u+)\,du\right\} \to_D V_k^{(2)},$$

where $V_m^{(1)}$ and $V_k^{(2)}$ are as in (1.9) *with $\psi_j(s) = 0$ for all $1 \leq s < \infty$, $j = 1, 2$. In particular, they are non-degenerate if $\psi_1 \not\equiv 0$ and $\psi_2 \not\equiv 0$, respectively, and*

$$V_m^{(1)} \quad \text{and} \quad V_k^{(2)} \text{ are independent.} \tag{1.16}$$

It will assist the presentation of our further results if temporarily we consider now the no-trimming case $m = k = 0$ and link the above limiting random variables with the classical theory of infinitely divisible laws. Let $Y_1, Y_2, \ldots$ be independent, exponentially distributed random variables with mean one and partial sums $S_n = Y_1 + \cdots + Y_n$, $n \geq 1$, and consider the associated left-continuous standard Poisson process $N(t) = \sum_{n=1}^{\infty} I(S_n < t)$, $t > 0$, where $I(\cdot)$ is the indicator function. Also, let $\{Y_k^{(1)}, k \geq 1\}$ and $\{Y_k^{(2)}, k \geq 1\}$ be two independent copies of the sequence $\{Y_k, k \geq 1\}$ with associated partial sums $S_n^{(j)}$, $n \geq 1$, and Poisson processes $N_j(\cdot)$, $j = 1, 2$.

THEOREM 3. (i) *If ψ is a non-decreasing, non-positive, right-continuous function on $(0, \infty)$ for which*

$$\int_{\varepsilon}^{\infty} \psi^2(s)\,ds < \infty \qquad \text{for all } \varepsilon > 0,$$

then

$$V_0 = \int_{S_1}^{\infty} (s - N(s))\,d\psi(s) + \int_1^{S_1} s\,d\psi(s) + \psi(1)$$

is a well-defined (almost surely finite) infinitely divisible random variable with characteristic function

$$\begin{aligned}\phi_0(t) = Ee^{itV_0} &= \exp\left(\int_0^{\infty}\left\{e^{it\psi(u)} - 1 - \frac{it\psi(u)}{1+\psi^2(u)}\right\}du + it\gamma\right)\\ &= \exp\left(\int_{-\infty}^{0}\left\{e^{itx} - 1 - \frac{itx}{1+x^2}\right\}dL(x) + it\gamma\right),\\ &\qquad\qquad -\infty < t < \infty,\end{aligned}$$

where

$$L(x) = \inf\{s: \psi(s) \geq x\}, \qquad -\infty < x < 0,$$

and

$$\gamma = \int_0^1 \frac{\psi(u)}{1 + \psi^2(u)}\, du - \int_1^\infty \frac{\psi^3(u)}{1 + \psi^2(u)}\, du.$$

If $\psi \not\equiv 0$ *on* $(0, \infty)$ *then* V_0 *is non-degenerate.*

(ii) *Given any infinitely divisible law with characteristic function*

$$\phi(t) = \exp\left(it\theta - \frac{1}{2}\sigma^2 t^2 + \int_{-\infty}^0 \left\{ e^{itx} - 1 - \frac{itx}{1 + x^2} \right\} dL(x) \right.$$
$$\left. + \int_0^\infty \left\{ e^{itx} - 1 - \frac{itx}{1 + x^2} \right\} dR(x) \right),$$
$$-\infty < t < \infty,$$

where $\theta \in (-\infty, \infty)$ *and* $\sigma \in [0, \infty)$ *are uniquely determined constants and* L *and* R *are uniquely determined left-continuous and right-continuous Lévy measures on* $(-\infty, 0)$ *and* $(0, \infty)$, *respectively, i.e.,* $L(\cdot)$ *and* $R(\cdot)$ *are nondecreasing functions with* $L(-\infty) = R(\infty) = 0$ *and*

$$\int_{-\varepsilon}^0 x^2\, dL(x) + \int_0^\varepsilon x^2\, dR(x) < \infty \qquad \text{for any } \varepsilon > 0,$$

the random variable $V_0^{(1)} + \sigma Z + V_0^{(2)} + \theta - \gamma_{12}$, *where for* $j = 1, 2$,

$$V_0^{(j)} = (-1)^{j+1}\left\{ \int_{S_1^{(j)}}^\infty \big(u - N_j(u)\big)\, d\psi_j(u) + \int_1^{S_1^{(j)}} u\, d\psi_j(u) + \psi_j(1) \right\},$$

and Z *is a standard normal random variable such that* $N_1(\cdot)$, $N_2(\cdot)$, *and* Z *are independent,*

$$\psi_1(u) = \inf\{x < 0: L(x) > u\}, \qquad 0 < u < \infty,$$
$$\psi_2(u) = \inf\{x < 0: -R(-x) > u\}, \qquad 0 < u < \infty,$$

$(\inf \varnothing = 0)$ *and, with*

$$\gamma_j = (-1)^{j+1}\left\{ \int_0^1 \frac{\psi_j(u)}{1 + \psi_j^2(u)}\, du - \int_1^\infty \frac{\psi_j^3(u)}{1 + \psi_j^2(u)}\, du \right\}, \qquad j = 1, 2,$$

$\gamma_{12} = \gamma_1 + \gamma_2$, *has characteristic function* ϕ.

Leaving aside the asymptotic fine structure of our lightly trimmed sums $S_n(m, k) = \sum_{j=m+1}^{n-k} X_{j,n}$, the first two theorems together give the following one except the form of the characteristic function.

THEOREM 4. *Let $m \geq 0$ and $k \geq 0$ be fixed integers. If the conditions of Theorem 1 or Theorem 2 are satisfied along $\{n_1\}$, then there exist constants $A_n > 0$ and B_n and a subsequence $\{n_2\}$ of $\{n_1\}$ such that*

$$A_{n_2}^{-1}\left\{\sum_{j=m+1}^{n_2-k} X_{j,n_2} - B_{n_2}\right\} \to_D V_{m,k}, \qquad \text{as } n_2 \to \infty,$$

where

$$V_{m,k} = V_{m,k}(\psi_1, \psi_2, \sigma) = V_m^{(1)} + N(0, \sigma^2) + V_k^{(2)},$$

$V_m^{(1)}$ and $V_k^{(2)}$ are as in (1.9), $0 \leq \sigma \leq 1$, and the three variables $V_m^{(1)}$, $N(0, \sigma^2)$, and $V_k^{(2)}$ are independent. Under either of the two conditions we may choose $B_n \equiv n\mu_{m,k}(n)$. Under the condition of Theorem 1 we may choose $A_n \equiv a(n)$ and under the condition of Theorem 2, we have $\sigma = 0$ and $\psi_j(s) = 0$ for $s \geq 1$, $j = 1, 2$. Furthermore,

$$E\exp(itV_{m,k}) = \exp\left(-\frac{\sigma^2 t^2}{2}\right)\phi_m^{(1)}(t)\phi_k^{(2)}(-t), \qquad -\infty < t < \infty,$$

where, for $j = 1, 2$ and $h = m, k$,

$$\begin{aligned}\phi_h^{(j)}(t) &= E\exp\left(itV_h^{(j)}\right)\\ &= \int_0^\infty \exp\left(\int_x^\infty \left\{e^{it\psi_j(v)} - 1 - \frac{it\psi_j(v)}{1 + \psi_j^2(v)}\right\} dv\right) e^{it\rho_h^{(j)}(x)} \frac{x^h}{h!} e^{-x}\, dx,\end{aligned}$$

with

$$\begin{aligned}\rho_h^{(j)}(x) = \psi_j(x) + \int_{1+x}^{1+h} \psi_j(s)\, ds + \int_x^{1+x} \frac{\psi_j(s)}{1 + \psi_j^2(s)}\, ds\\ - \int_{1+x}^\infty \frac{\psi_j^3(s)}{1 + \psi_j^2(s)}\, ds.\end{aligned}$$

$V_m^{(1)}$ is non-degenerate if $\psi_1 \not\equiv 0$ and $V_k^{(2)}$ is non-degenerate if $\psi_2 \not\equiv 0$.

We note that all these integrals make sense because

$$\int_\varepsilon^\infty \psi_j^2(s)\, ds < \infty \qquad \text{for all } \varepsilon > 0,\ j = 1, 2, \tag{1.17}$$

under the conditions of both Theorem 1 and 2, as proved in Lemma 2.5 below.

Now we turn to necessary conditions of convergence in distribution along a sequence of positive integers.

THEOREM 5. *Let $m \geq 0$ and $k \geq 0$ be fixed integers. If there exist two sequences of constants $A_n > 0$ and B_n and a sequence of positive integers $\{n_1\}$ such that*

$$A_{n_1}^{-1}\left\{\sum_{j=m+1}^{n_1-k} X_{j,n_1} - B_{n_1}\right\} \tag{1.18}$$

converges in distribution to a non-degenerate limit, then there exist a subsequence $\{n_2\}$ of $\{n_1\}$ and non-decreasing, non-positive, right-continuous functions ψ_1^* and ψ_2^* defined on $(0, \infty)$ such that

$$\int_\varepsilon^\infty \left(\psi_j^*(s)\right)^2 ds < \infty \qquad \textit{for all } \varepsilon > 0,\ j = 1, 2, \tag{1.19}$$

$$\frac{a(n_2)}{A_{n_2}}\psi_j(n_2, s) \to \psi_j^*(s), \qquad j = 1, 2, \tag{1.20}$$

at all continuity points s of ψ_1^* and ψ_2^*, respectively, and

$$a(n_2)/A_{n_2} \to \delta < \infty \tag{1.21}$$

as $n_2 \to \infty$, where δ is some non-negative constant. The limiting random variable of the sequence in (1.18) *is necessarily a linear function of the limiting random variable $V_{m,k}(\psi_1^*, \psi_2^*, \sigma)$ of Theorem* 4. *If $\delta > 0$ then either $\sigma > 0$, or at least one of ψ_1^* and ψ_2^* is not identically zero. If $\delta = 0$ then $\sigma = 0$ and $\psi_j^* = 0$ on $[1, \infty)$, $j = 1, 2$, but at least one of them is not identically zero.*

We note here that an examination of the proof of Theorem 5 shows the following: If there exists a subsequence $\{n_1\}$ such that

$$\left(a(n_1)\right)^{-1}\left\{\sum_{j=m+1}^{n_1-k} X_{j,n_1} - n_1\mu_{m,k}(n_1)\right\}$$

converges in distribution to a non-degenerate limit, then on the same subsequence

$$\limsup_{n_1\to\infty}\left|\psi_j(n_1, s)\right| < \infty, \qquad 0 < s < \infty,\ j = 1, 2. \tag{1.22}$$

In order to present the sequence of corollaries noted above in a compact fashion we need some further notation. For any choice of fixed non-negative integers m and k set $S_n(m, k) = \sum_{j=m+1}^{n-k} X_{j,n}$. We write $F \in D_p^{(m,k)}$ to mean that the (m, k)-trimmed sums $S_n(m, k)$ from the distribution F are in the domain of partial attraction of some non-degenerate random variable W, i.e. that there exist a subsequence $\{n_1\} \subset \{n\}$ (where $\{n\} = \{1, 2, \dots\}$) diverging to infinity and sequences of normalizing and centering constants $A_{n_1} > 0$ and B_{n_1} such that

$$A_{n_1}^{-1}\{S_{n_1}(m, k) - B_{n_1}\} \to_D W \qquad \text{as } n_1 \to \infty. \tag{1.23}$$

(Of course we know from Theorem 5 that W must have the distribution of a linear function of a random variable of the form $V_{m,k}(\psi_1, \psi_2, \sigma)$ given in Theorems 4 and 5.) Whenever the random variable W in (1.23) is a nondegenerate normal random variable we shall write $F \in D_p^{(m,k)}(2)$, and if in addition the sequence $\{n_1\}$ can be chosen to be the whole sequence $\{n\}$ we shall write $F \in D^{(m,k)}(2)$ to denote that the (m, k)-trimmed sums from the distribution F are in the domain of partial attraction or domain of attraction, respectively, of a normal law. If in the latter case the further additional constraint that A_n can be chosen as $A_n \equiv n^{1/2}$ is also satisfied, we write $F \in DN^{(m,k)}(2)$ to denote that the (m, k)-trimmed sums from the distribution F are in the domain of normal attraction of a normal law. When $m = k = 0$, that is when we talk about the whole untrimmed sum $S_n(0,0)$ being in the domain of partial attraction of some non-degenerate (necessarily infinitely divisible by Theorem 3) random variable, or in the domain of partial attraction of a non-degenerate normal, or in the domain of attraction of a non-degenerate normal, or in the domain of normal attraction of a non-degenerate normal, or in the domain of attraction of some non-degenerate random variable W, we just simply drop the superscript $(0,0)$ and write $F \in D_p$, $F \in D_p(2)$, $F \in D(2)$, $F \in DN(2)$, and $F \in D$, respectively. Similarly, if (1.23) holds with $m = k = 0$ and W being a non-degenerate, non-normal stable random variable of index $\alpha \in (0, 2)$, we write $F \in D_p(\alpha)$, if in addition this holds with $\{n_1\} = \{n\}$, we write $F \in D(\alpha)$, and if the latter holds with $A_n \equiv n^{1/\alpha}$, we write $F \in DN(\alpha)$ to denote that the whole sum $S_n(0,0)$ from F is in the domain of partial attraction, domain of attraction, and domain of normal attraction of this non-degenerate, non-normal stable variable with index α.

For a fixed choice of $m \geq 0$ and $k \geq 0$ the trimmed or whole sum $S_n(m, k)$ from the distribution F will be said to be stochastically compact if there exist sequences of normalizing and centering constants $A_n > 0$ and B_n such that for every subsequence $\{n_1\}$ of $\{n\}$ there exists a further subsequence $\{n_2\}$ of $\{n_1\}$ such that

$$A_{n_2}^{-1}\{S_{n_2}(m, k) - B_{n_2}\} \to_D W \qquad \text{as } n_2 \to \infty, \tag{1.24}$$

where W is a non-degenerate random variable (depending in general on $\{n_2\}$). This will be denoted by $F \in SC(m, k)$, and when $m = k = 0$, we shall write $F \in SC$.

For a fixed choice of $m \geq 0$ and $k \geq 0$, the trimmed or untrimmed sum $S_n(m, k)$ from the distribution F is called subsequentially stochastically compact if for every subsequence $\{n_1\}$ of $\{n\}$ there exists a further subsequence $\{n_2\}$ of $\{n_1\}$ and sequences of normalizing and centering constants $A_{n_2} > 0$ and B_{n_2} such that the convergence in distribution in (1.24) occurs for some non-degenerate random variable W. This we write as $F \in SSC(m, k)$ and if $m = k = 0$, as $F \in SSC$.

Recalling the truncated variance function $\sigma^2(s) = \sigma^2(s, 1-s)$ and introducing three other basic functions

$$H(s) := |Q(s+)| + |Q(1-s)|, \qquad 0 < s < 1,$$

$$G^2(s) := Q^2(s) + Q^2(1-s), \qquad 0 < s < 1,$$

and

$$S^2(s) := \int_s^{1-s} Q^2(u)\, du, \qquad 0 < s < \tfrac{1}{2},$$

we can now start stating the corollaries.

Corollary 1. *The following five statements are equivalent*:

$$\textit{For fixed } m \geq 0 \textit{ and } k \geq 0, \qquad F \in D^{(m,k)}(2); \tag{1.25}$$

$$\lim_{s \downarrow 0} s^{1/2} H(\lambda s)/\sigma(s) = 0 \qquad \textit{for all } 0 < \lambda < \infty; \tag{1.26a}$$

$$\lim_{s \downarrow 0} \sigma(\lambda s)/\sigma(s) = 1 \qquad \textit{for all } 0 < \lambda < 1; \tag{1.26b}$$

$$\lim_{s \downarrow 0} s G^2(s)/S^2(s) = 0; \tag{1.26c}$$

$$F \in D^{(m,k)}(2) \qquad \textit{for all } m \geq 0 \textit{ and } k \geq 0. \tag{1.27}$$

Corollary 2. *The following five statements are equivalent*:

$$\textit{For fixed } m \geq 0 \textit{ and } k \geq 0, \qquad F \in D_p^{(m,k)}(2); \tag{1.28}$$

$$\liminf_{s \downarrow 0} s^{1/2} H(\lambda s)/\sigma(s) = 0 \qquad \textit{for all } 0 < \lambda < \infty; \tag{1.29a}$$

There exists a sequence $s_m \in (0, 1)$ *with* $s_m \to 0$ *as* $m \to \infty$ *such that*

$$\lim_{m \to \infty} \sigma(\lambda s_m)/\sigma(s_m) = 1 \qquad \textit{for all } 0 < \lambda < 1; \tag{1.29b}$$

There exists a sequence $s_m \in (0,1)$ with $s_m \to 0$ as $m \to \infty$ such that

$$\lim_{m\to\infty} s_m G^2(\lambda s_m)/S^2(s_m) = 0 \qquad \textit{for all } 0 < \lambda < \infty; \tag{1.29c}$$

$$F \in D_p^{(m,k)}(2) \qquad \textit{for all } m \ge 0 \textit{ and } k \ge 0. \tag{1.30}$$

If one of these conditions is satisfied, for each $m \ge 0$ and $k \ge 0$ we have (1.4) *along some subsequence $\{n_1\}$.*

COROLLARY 3. *Assume $F \notin D(2)$. Then the following three statements are equivalent:*

$$F \in D; \tag{1.31}$$

There exist constants $0 < \alpha < 2$, $0 \le \delta_1, \delta_2 < \infty$, where at least one of the δ_1 and δ_2 is not zero, and a non-negative function L defined on $(0,1)$ and slowly varying at zero such that

$$-Q(s+) = s^{-1/\alpha}L(s)(\delta_1 + o(1)) \tag{1.32}$$

and

$$-Q(1-s) = s^{-1/\alpha}L(s)(\delta_2 + o(1)) \qquad \textit{as } s \downarrow 0; \tag{1.32}$$

$$F \in D(\alpha) \qquad \textit{for some } 0 < \alpha < 2. \tag{1.33}$$

If (1.32) *holds then for any fixed $m \ge 0$ and $k \ge 0$*

$$\frac{1}{n^{1/\alpha}L(1/n)}\left\{\sum_{j=m+1}^{n-k} X_{j,n} - n\mu_{m,k}(n)\right\} \to_D V_{m,k}\left(\psi_1^{(\alpha)}, \psi_2^{(\alpha)}, 0\right), \tag{1.34}$$

where this limiting random variable is as in Theorem 4 given by

$$\psi_j^{(\alpha)}(s) = -\left(\frac{2\left(\delta_1^2 + \delta_2^2\right)}{2-\alpha}\right)^{1/2} \delta_j s^{-1/\alpha}, \qquad 0 < s < \infty,\ j = 1,2.$$

Remark. We see that non-normal stable laws are given by the functions ψ_j of the form $\psi_j^{(\alpha)}(s) = -c_j s^{-1/\alpha}$, $0 < s < \infty$, $j = 1,2$, where the constants $c_1 \ge 0$, $c_2 \ge 0$, and α are such that $c_1 + c_2 > 0$ and $0 < \alpha < 2$. Using Theorem 3, a simple computation could easily identify the skewness, scale, and location parameters of the limiting stable random variable of exponent α in (1.34). This in turn would lead to a representation of a stable random variable with an arbitrary configuration of the four parameters in the form $V_{0,0}(\psi_1^{(\alpha)}, \psi_2^{(\alpha)}, 0) + d$, where the constants c_1 and c_2 in $\psi_1^{(\alpha)}$ and $\psi_2^{(\alpha)}$ and the constant d are determined by the given configuration. Then, writing $d\psi_j^{(\alpha)}(u) = c_j \alpha^{-1} u^{-1/\alpha - 1}\,du$ in the integrals, we would arrive at the representation of Theorem 3.2 in M. Csörgő, S. Csörgő, Horváth, and Mason [3]. (Here we use the occasion to correct some misprints and computational errors in this representation, on pages 107–108 of [3]: Just above formula (3.62), in the limiting characteristic function, $1 - 2p$ should be $2p - 1$; in formula (3.62) the factor $(\alpha - 1)^{-1}$ should be $(1 - \alpha)^{-1}$; the correct definition of $\theta(\alpha, p)$ when $\alpha = 1$ is $\theta(1, p) = A(2p - 1) - p \log p + (1 - p)\log(1 - p)$; and in Theorem 3.2, $\Delta_{\alpha,1}$ should be $-\Delta_{\alpha,1}$.)

COROLLARY 4. (i) *The following three statements are equivalent*: $F \in DN^{(m,k)}(2)$ *for fixed* $m \geq 0$ *and* $k \geq 0$; $F \in DN^{(m,k)}(2)$ *for all* $m \geq 0$ *and* $k \geq 0$; $EX^2 < \infty$.

(ii) $F \in DN(\alpha)$ *for a given* $\alpha \in (0,2)$ *if and only if* (1.32) *holds with the given* α *and* $L \equiv 1$. *In this case* (1.34) *is true with* $L(1/n) \equiv 1$.

COROLLARY 5. $F \in D_p(\alpha)$ *for a given* $\alpha \in (0,2)$ *if and only if there exist a subsequence* $\{n_1\}$ *of* $\{n\}$ *and constants* $c_1, c_2 \geq 0$ *such that* $c_1 + c_2 > 0$ *and*

$$\psi_j(n_1, s) \to \psi_j^{(\alpha)}(s) = -c_j s^{-1/\alpha}, \qquad 0 < s < \infty,\ j = 1,2,$$

and a further subsequence $\{n_2\} \subset \{n_1\}$ *such that*

$$\sigma^2(r_{n_2}/n_2)/\sigma^2(1/n_2) \to 0,$$

where $\{r_{n_1}\}$ *is the sequence of integers given by Theorem* 1. *In this case*

$$\frac{1}{a(n_2)}\left\{\sum_{j=m+1}^{n_2-k} X_{j,n_2} - n_2\mu_{m,k}(n_2)\right\} \to_D V_{m,k}\left(\psi_1^{(\alpha)}, \psi_2^{(\alpha)}, 0\right)$$

for each fixed $m \geq 0$ *and* $k \geq 0$. *Moreover, for any* $\alpha \in (0,2)$, $D(\alpha) \subset D_p(\alpha)$ *but* $D(\alpha) \neq D_p(\alpha)$.

COROLLARY 6. *There exist a sequence* $\{n_1\}$ *of positive integers and constants* $A_{n_1} > 0$ *and* B_{n_1} *along it such that for all fixed* $m \geq 1$ *and* $k \geq 1$

$$A_{n_1}^{-1}\left\{S_{n_1}(m,k) - B_{n_1}\right\} \to_D W_{m,k} \qquad \text{as } n_1 \to \infty \tag{1.35}$$

with non-degenerate random variables $W_{m,k}$ *if and only if*

$$A_{n_1}^{-1}\left\{S_{n_1}(0,0) - B_{n_1}\right\} \to_D W \qquad \text{as } n_1 \to \infty \tag{1.36}$$

with a non-degenerate random variable W. *In this case* $W_{m,k} = c_{m,k}V_{m,k}(\psi_1, \psi_2, \sigma) + d_{m,k}$ *with some constants* $c_{m,k}$ *and* $d_{m,k}$ *for all*

$m \geq 0$ and $k \geq 0$, where $V_{m,k}(\psi_1, \psi_2, \sigma)$ is the random variable described in Theorem 4, where ψ_1, ψ_2, and σ are independent of m and k and hence are uniquely determined by the limiting infinitely divisible law of W.

COROLLARY 7. *Assume $F \notin D_p(2)$. Then the following three statements are equivalent:*

$$\text{For fixed } m \geq 0 \text{ and } k \geq 0,\ F \in D_p^{(m,k)}; \tag{1.37}$$

There exist $0 < \lambda_0 < 1$ and a subsequence $\{n_1\}$ of $\{n\}$ such that

$$\limsup_{n_1 \to \infty} H(\lambda/n_1)/H(\lambda_0/n_1) < \infty \qquad \text{for all } 0 < \lambda < \infty; \tag{1.38}$$

$$F \in D_p^{(m,k)} \qquad \text{for all } m \geq 0 \text{ and } k \geq 0. \tag{1.39}$$

If condition (1.38) is satisfied we can always choose $A_{n_2} \equiv n_2^{1/2} H(\lambda_0/n_2)$ and $B_{n_2} \equiv n_2 \mu_{m,k}(n_2)$, and with this choice (1.24) holds true along some $\{n_2\} \subset \{n_1\}$.

COROLLARY 8. *For any (non-degenerate) distribution function the five statements (1.37), (1.39), and*

There exists a subsequence $\{n_1\}$ of $\{n\}$ such that

$$\limsup_{n_1 \to \infty} n_1^{-1/2} H(\lambda/n_1)/\sigma(1/n_1) < \infty \qquad \text{for all } 0 < \lambda < \infty; \tag{1.40a}$$

There exists a sequence $s_m \in (0,1)$ with $s_m \to 0$ as $m \to \infty$ such that

$$\limsup_{m \to \infty} \sigma(\lambda s_m)/\sigma(s_m) < \infty \qquad \text{for all } 0 < \lambda < 1; \tag{1.40b}$$

There exists a subsequence $\{n_1\}$ of $\{n\}$ such that

$$\limsup_{n_1 \to \infty} n_1^{-1} G^2(\lambda/n_1)/S^2(1/n_1) < \infty \qquad \text{for all } 0 < \lambda < \infty; \tag{1.40c}$$

are equivalent.

The condition (1.29b) gives an interesting interpretation of the sequence $\{r_{n_2}\}$ of Theorem 1. From (1.10) it is easy to infer that when $0 < \sigma \leq 1$, one has for all $0 < \lambda < 1$,

$$\lim_{n_2 \to \infty} \sigma\big(\lambda(r_{n_2} + 1)/n_2\big)/\sigma\big((r_{n_2} + 1)/n_2\big) = 1,$$

which by (1.29b) implies that $F \in D_p(2)$. Thus from Theorems 1 and 5 we conclude the following.

COROLLARY 9. *F is in the domain of partial attraction of an infinitely divisible law with a non-degenerate normal component if and only if* $F \in D_p(2)$.

Now we turn to stochastic compactness.

COROLLARY 10. *The following five statements are equivalent:*

$$\text{For fixed } m \geq 0 \text{ and } k \geq 0, \qquad F \in SC(m, k); \tag{1.41}$$

$$\limsup_{s \downarrow 0} s^{1/2} H(\lambda s)/\sigma(s) < \infty \qquad \text{for all } 0 < \lambda < \infty; \tag{1.42a}$$

$$\limsup_{s \downarrow 0} \sigma(\lambda s)/\sigma(s) < \infty \qquad \text{for all } 0 < \lambda < 1; \tag{1.42b}$$

$$\limsup_{s \downarrow 0} s G^2(s)/S^2(s) < \infty; \tag{1.42c}$$

$$F \in SC(m, k) \qquad \text{for all } m \geq 0 \text{ and } k \geq 0. \tag{1.43}$$

If one of these conditions is satisfied we can always choose $A_n \equiv a(n) = n^{1/2}\sigma(1/n)$ *and, for a given* $m \geq 0$ *and* $k \geq 0$, $B_n \equiv n\mu_{m,k}(n)$ *as normalizing and centering constants, and the functions* ψ_1 *and* ψ_2 *and the constant* σ, *arising in any subsequential limiting random variable* $cV_{m,k}(\psi_1, \psi_2, \sigma) + d$, *necessarily satisfy the inequality*

$$s\{\psi_1^2(s) + \psi_2^2(s)\} \leq C\left\{\sigma^2 + \int_s^\infty \left(\psi_1^2(t) + \psi_2^2(t)\right) dt\right\} \tag{1.44}$$

for $0 < s < \infty$, *where C is a finite constant depending only on F and the chosen normalizing sequence.*

COROLLARY 11. *The following three statements are equivalent:*

$$\text{For fixed } m \geq 0 \text{ and } k \geq 0, \qquad F \in SSC(m, k). \tag{1.45}$$

For every subsequence $\{n_1\}$ *of* $\{n\}$ *there exist a further subsequence* $\{n_2\} \subset \{n_1\}$, *a sequence of numbers* $A_{n_2} > 0$ *and non-decreasing, non-positive, right-continuous functions* ψ_1^* *and* ψ_2^* *such that* (1.19), (1.20), *and* (1.21) *are satisfied.* (1.46)

$$F \in SSC(m, k) \qquad \text{for all } m \geq 0 \text{ and } k \geq 0. \tag{1.47}$$

Moreover, the class of all distributions such that $F \in SSC$ but $F \notin SC$ is not empty.

The last corollary links the stochastic compactness of sums $S_n(0,0)$ and the stochastic compactness of extreme values. To be definite, we formulate it for the maxima $X_{n,n} = \max(X_1, \ldots, X_n)$ and for the sake of simplicity we assume that the random variables are not negative. We call $\{X_{n,n}\}$ stochastically compact if there exists a sequence $A_n > 0$ such that $\{A_n^{-1} X_{n,n}\}$ is stochastically compact, i.e., for each subsequence $\{n_1\}$ of $\{n\}$ there exists a further subsequence $\{n_2\}$ of $\{n_1\}$ such that $A_{n_2}^{-1} X_{n_2, n_2}$ converges in distribution to a non-degenerate random variable as $n_2 \to \infty$.

COROLLARY 12. *Suppose that $F(0-) = 0$ and $F \notin D_p(2)$. Then $F \in SC$ if and only if $\{X_{n,n}\}$ is stochastically compact. In this case the normalizing constants can always be chosen as $A_n \equiv a(n)$ and the subsequential limiting random variables of $\{X_{n,n}/a(n)\}$ are necessarily of the form $\phi(Y)$, where Y is a mean-one exponential random variable and ϕ is a non-negative, non-increasing, right-continuous function on $(0, \infty)$ such that $\int_\varepsilon^\infty \phi^2(t)\,dt < \infty$ for all $\varepsilon > 0$ and*

$$s\phi^2(s) \le C \int_s^\infty \phi^2(t)\,dt, \qquad 0 < s < \infty,$$

with some finite constant C depending on F.

2. PROOFS

We shall be working on a specially constructed probability space (Ω, A, P) described in [2, 3]. It carries two independent sequences $\{Y_n^{(j)}, n \ge 1\}$, $j = 1, 2$, of independent, exponentially distributed random variables with mean 1 and a sequence $\{B_n(t), 0 \le t \le 1; n \ge 1\}$ of Brownian bridges with the following property: For each $n \ge 2$, let

$$Y_j(n) = \begin{cases} Y_j^{(1)}, & j = 1, \ldots, [n/2], \\ Y_{n+2-j}^{(2)}, & j = [n/2] + 1, \ldots, n+1, \end{cases}$$

and for $m = 1, \ldots, n+1$, write

$$S_m(n) = \sum_{j=1}^{m} Y_j(n).$$

Then the ratios $U_{k,n} = S_k(n)/S_{n+1}(n)$, $k = 1, \dots, n$, have the same joint distribution as the order statistics of n independent uniform (0,1) random variables and for their left-continuous empirical distribution function

$$G_n^{(1)}(u) = n^{-1} \sum_{j=1}^{n} I\big(U_{j,n} < u\big), \qquad 0 \le u \le 1,$$

we have

$$\Delta(n) = \sup_{n^{-1} \le t \le 1 - n^{-1}} n^{\nu} \frac{|n^{1/2}\big(G_n^{(1)}(t) - t\big) - B_n(t)|}{\big(t(1-t)\big)^{1/2-\nu}} = O_p(1) \quad (2.1)$$

as $n \to \infty$ for any fixed $\nu \in [0, \frac{1}{4})$. We shall also need

$$G_n^{(2)}(u) = n^{-1} \sum_{j=1}^{n} I\big(1 - U_{n+1-j,n} < u\big), \qquad 0 \le u \le 1,$$

and the independent standard left-continuous Poisson processes

$$N_j(s) = \sum_{k=1}^{\infty} I\big(S_k^{(j)} < s\big), \qquad 0 \le s < \infty,\ j = 1,2,$$

associated with the two independent partial sum sequences $\{S_k^{(j)} = Y_1^{(j)} + \cdots + Y_k^{(j)}, k \ge 1\}$. Since we are only interested in distributional properties, in view of (1.1) we shall write with some abuse of notation

$$\big(X_{1,n}, \dots, X_{n,n}\big) = \big(Q(U_{1,n}), \dots, Q(U_{n,n})\big), \qquad n \ge 2. \quad (2.2)$$

For the proofs of Theorems 1 and 2 it is convenient to deal with the two conditions (1.3) and (1.11) simultaneously. Therefore, we assume that there exists a subsequence $\{n_1\}$ of the positive integers such that for a sequence of positive constants C_n and two non-decreasing, right-continuous functions ψ_1 and ψ_2 we have

$$C_{n_1}\psi_j(n_1, s) \to \psi_j(s) \qquad \text{as } n_1 \to \infty \quad (2.3)$$

at every continuity point $s \in (0, \infty)$ of ψ_j, $j = 1, 2$.

Fix the integers l, r, m, and k such that

$$m \le l \le r < n - r \le n - l \le n - k \quad (2.4)$$

and write

$$\begin{aligned}
T_{m,k}(n) &:= \frac{C_n}{a(n)}\left\{\sum_{j=m+1}^{n-k} X_{j,n} - n\int_{(m+1)/n}^{1-(k+1)/n} Q(u+)\,du\right\} \\
&= \frac{C_n}{a(n)}\left\{\sum_{j=m+1}^{l} X_{j,n} - n\int_{(m+1)/n}^{(l+1)/n} Q(u+)\,du\right\} \\
&\quad + \frac{C_n}{a(n)}\left\{\sum_{j=l+1}^{r} X_{j,n} - n\int_{(l+1)/n}^{(r+1)/n} Q(u+)\,du\right\} \\
&\quad + \frac{C_n}{a(n)}\left\{\sum_{j=r+1}^{n-r} X_{j,n} - n\int_{(r+1)/n}^{1-(r+1)/n} Q(u+)\,du\right\} \\
&\quad + \frac{C_n}{a(n)}\left\{\sum_{j=n-r+1}^{n-l} X_{j,n} - n\int_{1-(r+1)/n}^{1-(l+1)/n} Q(u+)\,du\right\} \\
&\quad + \frac{C_n}{a(n)}\left\{\sum_{j=n-l+1}^{n-k} X_{j,n} - n\int_{1-(l+1)/n}^{1-(k+1)/n} Q(u+)\,du\right\} \\
&=: V_m^{(1)}(l,n) + \Delta_1(l,r,n) + M(r,n) \\
&\quad + \Delta_2(l,r,n) + V_k^{(2)}(l,n). \qquad (2.5)
\end{aligned}$$

We note that the inequalities in (2.4) will not restrict the choice of the fixed $m \geq 0$ and $k \geq 0$ because l will be going to infinity later. Throughout we shall use the same integral convention as in [3]. Using this convention and the convention in (2.2) we have

$$V_m^{(1)}(l,n) = \frac{nC_n}{a(n)}\left\{\int_{U_{m+1,n}}^{U_{l+1,n}} Q(u+)\,dG_n^{(1)}(u) - \int_{(m+1)/n}^{(l+1)/n} Q(u+)\,du\right\}$$

and

$$V_k^{(2)}(l,n) = \frac{nC_n}{a(n)}\left\{\int_{1-U_{n-k,n}}^{1-U_{n-l,n}} Q(1-u)\,dG_n^{(2)}(u) - \int_{(k+1)/n}^{(l+1)/n} Q(1-u)\,du\right\}$$

First we deal with these two extreme terms.

Proposition 2.1. *If* (2.3) *holds then for any fixed* $l \geq 1$ *satisfying* (2.4) *we have*

$$V_m^{(1)}(l, n_1) \to_p V_m^{(1)}(l)$$
$$:= \int_{S_{m+1}^{(1)}}^{S_{l+1}^{(1)}} (u - N_1(u))\, d\psi_1(u) + \int_{m+1}^{S_{m+1}^{(1)}} (u - (m+1))\, d\psi_1(u)$$
$$+\psi_1(S_{m+1}^{(1)}) - \int_{l+1}^{S_{l+1}^{(1)}} (u - (l+1))\, d\psi_1(u) - \psi_1(S_{l+1}^{(1)})$$

and

$$V_k^{(2)}(l, n_1) \to_p V_k^{(2)}(l)$$
$$:= -\left\{ \int_{S_{k+1}^{(2)}}^{S_{l+1}^{(2)}} (u - N_2(u))\, d\psi_2(u) + \int_{k+1}^{S_{k+1}^{(2)}} (u - (k+1))\, d\psi_2(u) \right.$$
$$\left. +\psi_2(S_{k+1}^{(2)}) - \int_{l+1}^{S_{l+1}^{(2)}} (u - (l+1))\, d\psi_2(u) - \psi_2(S_{l+1}^{(2)}) \right\}$$

as $n_1 \to \infty$.

The proof of Proposition 2.1 will be given in a sequence of three lemmas. For $n \geq 2$ and integers $q = 1, \ldots, n$ we set

$$Z_{q,n}^{(j)} = \begin{cases} nU_{q,n}, & j = 1, \\ n(1 - U_{n+1-q,n}), & j = 2. \end{cases}$$

For $j = 1, 2$ and $h = m, k$ an integration by parts and a change of variables yield that on the set

$$F_n = \{\alpha_n \leq U_{1,n} \leq U_{n,n} \leq 1 - \alpha_n\}$$

of (1.2) we have

$$V_h^{(j)}(l, n) = (-1)^{j+1} \left\{ \int_{Z_{h+1,n}^{(j)}}^{Z_{l+1,n}^{(j)}} \left(u - nG_n^{(j)}(u/n)\right) dC_n \psi_j(n, u) \right.$$
$$+ \int_{h+1}^{Z_{h+1,n}^{(j)}} (u - (h+1))\, dC_n \psi_j(n, u)$$
$$+ \int_{Z_{l+1,n}^{(j)}}^{l+1} (u - (l+1))\, dC_n \psi_j(n, u)$$
$$\left. + C_n \psi_j\left(n, Z_{h+1,n}^{(j)}\right) - C_n \psi_j\left(n, Z_{l+1,m}^{(j)}\right) \right\}$$
$$=: (-1)^{j+1} \{ D_h^{(j)}(l, n) + E_h^{(j)}(n)$$
$$- E_l^{(j)}(n) + F_h^{(j)}(n) - F_l^{(j)}(n) \}.$$

LEMMA 2.1. *If* (2.3) *holds then for* $j = 1, 2$ *and each integer* $h \geq 0$,

$$F_h^{(j)}(n_1) = C_{n_1}\psi_j\left(n_1, Z_{h+1,n_1}^{(j)}\right) \to_p \psi_j\left(S_{h+1}^{(j)}\right) \qquad \text{as } n_1 \to \infty \quad (2.6)$$

and

$$C_{n_1}\psi_j\left(n_1, Z_{h+1,n_1}^{(j)}\right) - C_{n_1}\psi_j\left(n_1, S_{h+1}^{(j)}\right) \to_p 0 \qquad \text{as } n_1 \to \infty. \quad (2.7)$$

Proof. By the construction of the underlying space (Ω, A, P) we have

$$Z_{h+1,n}^{(j)} \to_p S_{h+1}^{(j)} \qquad \text{as } n \to \infty. \quad (2.8)$$

Therefore, for any subsequence $\{n_2\}$ of $\{n_1\}$ there exists a further subsequence $\{n_3\}$ of $\{n_2\}$ such that

$$Z_{h+1,n_3}^{(j)} \to S_{h+1}^{(j)} \text{ a.s.} \qquad \text{as } n_3 \to \infty.$$

Since $S_{h+1}^{(j)}$ is almost surely a continuity point of ψ_j, we conclude from (2.3) that

$$C_{n_3}\psi_j\left(n_3, Z_{h+1,n_3}^{(j)}\right) \to \psi_j\left(S_{h+1}^{(j)}\right) \text{ a.s.} \qquad \text{as } n_3 \to \infty \quad (2.9)$$

and

$$C_{n_3}\psi_j\left(n_3, S_{h+1}^{(j)}\right) \to \psi_j\left(S_{h+1}^{(j)}\right) \text{ a.s.} \qquad \text{as } n_3 \to \infty.$$

Consequently, for any subsequence $\{n_2\}$ of $\{n_1\}$ there is a subsequence $\{n_3\}$ of $\{n_2\}$ such that (2.9) holds and

$$C_{n_3}\psi_j\left(n_3, Z_{h+1,n_3}^{(j)}\right) - C_{n_3}\psi_j\left(n_3, S_{h+1}^{(j)}\right) \to 0 \text{ a.s.} \qquad \text{as } n_3 \to \infty.$$

These two statements prove (2.6) and (2.7). □

LEMMA 2.2. *If* (2.3) *holds then for* $j = 1, 2$ *and* $h = m, k$ *and all integers* $l \geq 1$,

$$D_h^{(j)}(l, n_1) \to_p D_h^{(j)}(l) := \int_{S_{h+1}^{(j)}}^{S_{l+1}^{(j)}} \left(u - N_j(u)\right) d\psi_j(u) \qquad \text{as } n_1 \to \infty.$$

Proof. Let $j = 1, 2$, $h = m, k$ and $l \geq 1$ be fixed. We write

$$\begin{aligned}
&\left|D_h^{(j)}(l, n_1) - D_h^{(j)}(l)\right| \\
&\quad\leq \left|\int_{Z_{h+1,n_1}^{(j)}}^{S_{h+1}^{(j)}} \left(u - n_1 G_{n_1}^{(j)}(u/n_1)\right) dC_{n_1}\psi_j(n_1, u)\right| \\
&\qquad + \left|\int_{S_{l+1}^{(j)}}^{Z_{l+1,n_1}^{(j)}} \left(u - n_1 G_{n_1}^{(j)}(u/n_1)\right) dC_{n_1}\psi_j(n_1, u)\right| \\
&\qquad + \left|\int_{S_{h+1}^{(j)}}^{S_{l+1}^{(j)}} u\, dC_{n_1}\psi_j(n_1, u) - \int_{S_{h+1}^{(j)}}^{S_{l+1}^{(j)}} u\, d\psi_j(u)\right| \\
&\qquad + \left|\int_{S_{h+1}^{(j)}}^{S_{l+1}^{(j)}} N_j(u)\, d\psi_j(u) - \int_{S_{h+1}^{(j)}}^{S_{l+1}^{(j)}} N_j(u)\, dC_{n_1}\psi_j(n_1, u)\right| \\
&\qquad + \left|\int_{S_{h+1}^{(j)}}^{S_{l+1}^{(j)}} \left(N_j(u) - n_1 G_{n_1}^{(j)}(u/n)\right) dC_{n_1}\psi_j(n_1, u)\right| \\
&\quad =: \mathrm{I}(n_1) + \mathrm{II}(n_1) + \mathrm{III}(n_1) + \mathrm{IV}(n_1) + \mathrm{V}(n_1).
\end{aligned}$$

Then

$$\begin{aligned}
\mathrm{I}(n_1) \leq &\left\{S_{h+1}^{(j)} + n_1 G_{n_1}^{(j)}\left(S_{h+1}^{(j)}/n_1\right) + Z_{h+1,n_1}^{(j)} + n_1 G_{n_1}^{(j)}\left(Z_{h+1,n_1}^{(j)}/n_1\right)\right\} \\
&\times\left|C_{n_1}\psi_j\left(n_1, S_{h+1}^{(j)}\right) - C_{n_1}\psi_j\left(n_1, Z_{h+1,n_1}^{(j)}\right)\right|.
\end{aligned}$$

It is easy to see that the first factor on the right side is $O_p(1)$ as $n_1 \to \infty$. Hence by (2.7) we have $\mathrm{I}(n_1) = o_p(1)$ as $n_1 \to \infty$. By the same argument we also get $\mathrm{II}(n_1) = o_p(1)$ as $n_1 \to \infty$. Clearly, $\mathrm{III}(n_1) \to 0$ a.s. as $n_1 \to \infty$ by assumption (2.3) and the fact that $S_{h+1}^{(j)}$ and $S_{l+1}^{(j)}$ are continuity points of ψ_j with probability 1. Next we consider the fourth term. Since

$$\begin{aligned}
\mathrm{IV}(n_1) = \Bigg|\sum_{q=h+2}^{l+1} N_j\left(S_q^{(j)}\right)\Big\{&\psi_j\left(S_q^{(j)}\right) - C_{n_1}\psi_j\left(n_1, S_q^{(j)}\right) \\
&-\psi_j\left(S_{q-1}^{(j)}\right) + C_{n_1}\psi_j\left(n_1, S_{q-1}^{(j)}\right)\Big\}\Bigg|,
\end{aligned}$$

and $S_q^{(j)}$ and $S_{q-1}^{(j)}$ are a.s. continuity points of ψ_j, (2.3) implies again that $\mathrm{IV}(n_1) = o_p(1)$ as $n_1 \to \infty$. Finally, for the last term, we fix an arbitrary $T \in (0, \infty)$. On the event $\{S_{l+1}^{(j)} < T\}$ we have

$$\begin{aligned}
V(n_1) &\leq 2 \sup_{0<t<T}\left|\int_0^t \left(N_j(u) - n_1 G_{n_1}^{(j)}(u/n_1)\right) dC_{n_1}\psi_j(n_1, u)\right| \\
&=: 2V(n_1, T).
\end{aligned}$$

Since $P\{S_{l+1}^{(j)} < T\} \to 1$ as $T \to \infty$, it is enough to show that for each $T \in (0, \infty)$,

$$V(n_1, T) \to_p 0 \qquad \text{as } n_1 \to \infty. \tag{2.10}$$

For this we use the arguments of the proof of Lemma 3.4, pages 97–98, in [3]. Observe that the first part of the proof down to Eq. (3.25) is concerned with the behavior of the jump times of N_j only and has nothing to do with the measures $d\mu_{\alpha,n}(s)$ appearing in the formulation of the lemma. Furthermore, the argument on pages 98–99 of the same paper establishing

$$\left|\int_0^t \left(N^{(i)}(s) - \Gamma_n^{(i)}(s)\right) d\mu_{\alpha,n}(s)\right| \le \sum_{j=1}^{m_i+1} \int_{I_j^{(i)}(\lambda)} d\mu_{\alpha,n}(s)$$

on the event $E_n^{(i)}$ for $0 < t < T$ is again independent of the measure $d\mu_{\alpha,n}(s)$ and can be used in the present situation to obtain

$$V(n_1, T) \le \sum_{q=1}^{m_j+1} \int_{I_q^{(j)}(\lambda)} dC_{n_1} \psi_j(n_1, u)$$

on the event $E_{n_1}^{(j)}$. Since $I_q^{(j)}(\lambda) = (S_q^{(j)}/\lambda, \lambda S_q^{(j)}]$, $\lambda > 1$, and since $S_q^{(j)}/\lambda$ and $\lambda S_q^{(j)}$ are continuity points of ψ_j with probability 1, we conclude from (2.3) that

$$\sum_{q=1}^{m_j+1} \int_{I_q^{(j)}(\lambda)} dC_{n_1} \psi_j(n_1, u) \to \sum_{q=1}^{m_j+1} \left\{\psi_j\left(\lambda S_q^{(j)}\right) - \psi_j\left(S_q^{(j)}/\lambda\right)\right\} \text{ a.s.}$$

as $n_1 \to \infty$. Since $S_q^{(j)}$ is also a.s. a continuity point of ψ_j, we see that

$$\sum_{q=1}^{m_j+1} \left\{\psi_j\left(\lambda S_q^{(j)}\right) - \psi_j\left(S_q^{(j)}/\lambda\right)\right\} \to 0 \text{ a.s.} \qquad \text{as } \lambda \downarrow 1.$$

On account of the fact that $P\{E_{n_1}^{(j)}\} \to 1$ as $n_1 \to \infty$, this entails (2.10) and completes the proof of the lemma. □

LEMMA 2.3. *If* (2.3) *holds then for each integer* $h \ge 0$ *and* $j = 1, 2$,

$$E_h^{(j)}(n_1) \to_p \int_{h+1}^{S_{h+1}^{(j)}} (u - (h+1)) \, d\psi_j(u) \qquad \text{as } n_1 \to \infty. \tag{2.11}$$

Proof. Fix $j = 1, 2$ and let $\bar{c}_h^{(j)}$ and $c_h^{(j)}$ be continuity points of ψ_j such that $0 < \bar{c}_h^{(j)} \le h + 1 \le c_h^{(j)}$. Writing

$$\begin{aligned} E_h^{(j)}(n_1) &= \int_{h+1}^{c_h^{(j)}} (u - (h+1))\, dC_{n_1} \psi_j(n_1, u) \\ &\quad + \int_{c_h^{(j)}}^{Z_{h+1,n_1}^{(j)}} (u - (h+1))\, dC_{n_1} \psi_j(n_1, u) \\ &=: \mathrm{I}(n_1) + \mathrm{II}(n_2), \end{aligned}$$

assumption (2.3), through (2.7) and an argument like that used to handle the first two terms in the proof of Lemma 2.2 implies

$$\mathrm{II}(n_1) \to_p \int_{c_h^{(j)}}^{S_{h+1}^{(j)}} (u - (h+1))\, d\psi_j(u) \qquad \text{as } n_1 \to \infty$$

for $S_{h+1}^{(j)}$ is also a continuity point of ψ_j, with probability one. Furthermore,

$$\begin{aligned} |\mathrm{I}(n_1)| &\le \int_{\bar{c}_h^{(j)}}^{c_h^{(j)}} |u - (h+1)|\, dC_{n_1} \psi_j(n_1, u) \\ &\le \left(c_h^{(j)} - \bar{c}_h^{(j)}\right) C_{n_1} \left\{ \psi_j\left(n_1, c_h^{(j)}\right) - \psi_j\left(n_1, \bar{c}_h^{(j)}\right) \right\} \\ &\to \left(c_h^{(j)} - \bar{c}_h^{(j)}\right)\left(\psi_j\left(c_h^{(j)}\right) - \psi_j\left(\bar{c}_h^{(j)}\right)\right) \end{aligned}$$

as $n_1 \to \infty$. Since $c_h^{(j)}$ and $\bar{c}_h^{(j)}$ can be chosen arbitrarily close to $h + 1$, the two limit relations easily give the lemma. □

Proposition 2.1 is obviously a consequence of Lemmas 2.1, 2.2, and 2.3.

Our next aim is to show that the functions ψ_1 and ψ_2 appearing in (1.3) and (1.11) are square integrable on $[\varepsilon, \infty)$ for each $\varepsilon > 0$. For this we need Lemma 2.1 of S. Csörgő, Haeusler, and Mason [4] which we state here as

LEMMA 2.4. *For any (nondegenerate) quantile function Q,*

$$\limsup_{s, t \downarrow 0} \left\{ tQ^2(1 - t) + sQ^2(s) \right\} / \sigma^2(s, 1 - t) < \infty.$$

LEMMA 2.5. *If* (2.3) *holds with either $C_{n_1} = 1$ for all n_1, or $C_{n_1} \to 0$ as $n_1 \to \infty$, then* (1.17) *also holds true.*

Proof. In a first step we establish that for all $s \in (1, \infty)$ and $j = 1, 2$,

$$\int_s^\infty \int_s^\infty (u \wedge v)\, d\psi_j(u)\, d\psi_j(v) < \infty. \tag{2.12}$$

Let $1 < s < t < \infty$ be arbitrary. Fix continuity points s' and t' of ψ_j such that $1 < s' < s < t < t' < \infty$. The arguments used in the proof of Lemma

2.2 give, as $n_1 \to \infty$,

$$\int_{s'}^{t'} \left(u - n_1 G_{n_1}^{(j)}(u/n_1)\right) dC_{n_1} \psi_j(n_1, u) \to_p \int_{s'}^{t'} \left(u - N_j(u)\right) d\psi_j(u),$$

so that there exists a subsequence $\{n_2\}$ of $\{n_1\}$ such that

$$\int_{s'}^{t'} \left(u - n_2 G_{n_2}^{(j)}(u/n_2)\right) dC_{n_2} \psi_j(n_2, u) \to \int_{s'}^{t'} \left(u - N_j(u)\right) d\psi_j(u) \text{ a.s.}$$

as $n_2 \to \infty$. Then we have

$$\begin{aligned}\int_s^t \int_s^t (u \wedge v)\, d\psi_j(u)\, d\psi_j(v) &\le \int_{s'}^{t'} \int_{s'}^{t'} (u \wedge v)\, d\psi_j(u)\, d\psi_j(v)\\ &= E\left(\int_{s'}^{t'} \left(u - N_j(u)\right) d\psi_j(u)\right)^2,\end{aligned}$$

which by Fatou's lemma is less than or equal to

$$\begin{aligned}&\liminf_{n_2 \to \infty} E\left(\int_{s'}^{t'} \left(u - n_2 G_{n_2}^{(j)}(u/n_2)\right) dC_{n_2} \psi_j(n_2, u)\right)^2\\ &\quad \le C \liminf_{n_2 \to \infty} \int_{s'/n_2}^{t'/n_2} \int_{s'/n_2}^{t'/n_2} (u \wedge v - uv)\, d\tilde{\psi}_j(u)\, d\tilde{\psi}_j(v)/\sigma^2(1/n_2) \le C,\end{aligned}$$

where $C < \infty$ is a bound for the convergent sequence $\{C_{n_1}^2, n_1 \ge 1\}$ and $\tilde{\psi}_1(u) = Q(u+)$, $\tilde{\psi}_2(u) = Q(1-u)$. Thus (2.12) is proven.

Now observe that from the definition of the $\psi_j(n, \cdot)$ functions, our assumptions on $\{C_{n_1}\}$, and Lemma 2.4 it follows that

$$\limsup_{t \to \infty} t^{1/2} |\psi_j(t)| < \infty, \qquad j = 1, 2,$$

so that

$$\psi_j(t) \to 0 \qquad \text{as } t \to \infty,\ j = 1, 2. \tag{2.13}$$

The relations in (2.12) and (2.13) imply (1.17). □

As we will let l of Proposition 2.1 go to infinity, we need the following.

LEMMA 2.6. *If* (2.3) *holds then for* $j = 1, 2$,

$$\psi_j\left(S_{l+1}^{(j)}\right) \to_p 0 \qquad \text{as } l \to \infty \tag{2.14}$$

and

$$\int_{S_{l+1}^{(j)}}^{l+1} (u - (l+1))\, d\psi_j(u) \to_p 0 \qquad \text{as } l \to \infty. \tag{2.15}$$

Proof. Statement (2.14) is immediate from (2.13) and the fact that $S_{l+1}^{(j)} \to \infty$ a.s. as $l \to \infty$. For the proof of (2.15) first observe that by the central limit theorem for each $c \in (0, \infty)$ we have

$$P\{A_c(l)\} \to 2\Phi(c) - 1 \qquad \text{as } l \to \infty, \tag{2.16}$$

where $A_c(l) = \{l + 1 - cl^{1/2} \le S_{l+1}^{(j)} \le l + 1 + cl^{1/2}\}$ and Φ here is the standard normal distribution function. On this event $A_c(l)$ we have

$$\left| \int_{S_{l+1}^{(j)}}^{l+1} (u - (l+1))\, d\psi_j(u) \right| \\ \le cl^{1/2}\{\psi_j(l + 1 + cl^{1/2}) - \psi_j(l + 1 - cl^{1/2})\},$$

where the right side converges to zero as $l \to \infty$ due to the fact that Lemma 2.5 implies

$$t^{1/2}\psi_j(t) \to 0 \qquad \text{as } t \to \infty,\ j = 1,2. \tag{2.17}$$

Combined with (2.16) this yields (2.15) upon letting $c \to \infty$. □

Next we deal with the middle part $M(r, n)$ in the decomposition (2.5).

PROPOSITION 2.2. *If (2.3) holds with either $C_{n_1} = 1$ for all $n_1 \ge 1$ or with $C_{n_1} \to 0$ as $n_1 \to \infty$, then for each fixed integer $r \ge 1$ and all $n_1 \ge 2(r + 1)$ we have*

$$M(r, n_1) = C_{n_1} \frac{\sigma((r+1)/n_1, 1 - (r+1)/n_1)}{\sigma(1/n_1, 1 - 1/n_1)} N_{r, n_1}(0,1) + R(r, n_1),$$

where $N_{r,n_1}(0,1)$ is a standard normal random variable for each n_1 and r and

$$\lim_{r \to \infty} \limsup_{n_1 \to \infty} P\{|R(r, n_1)| \ge \varepsilon\} = 0 \qquad \textit{for each } \varepsilon > 0. \tag{2.18}$$

The proof of Proposition 2.2 will be given in another sequence of four lemmas. By (2.2) we can write

$$M(r, n) = \frac{nC_n}{a(n)} \left\{ \int_{U_{r+1,n}}^{U_{n+1-r,n}} Q(u+)\, dG_n^{(1)}(u) - \int_{(r+1)/n}^{1-(r+1)/n} Q(u+)\, du \right\}.$$

From this, integrating by parts, rearranging terms, and using that $Q(U_{j,n}) = Q(U_{j,n}+)$, $1 \leq j \leq n$, with probability one, we obtain

$$\begin{aligned} M(r,n) &= \frac{C_n}{\sigma(1/n)}\left\{\int_{((r+1)/n,1-(r+1)/n)} n^{1/2}\big(u - G_n^{(1)}(u)\big)\, dQ(u)\right\} \\ &\quad + \frac{nC_n}{a(n)}\int_{(r+1)/n}^{U_{r+1,n}} \big(G_n^{(1)}(u) - r/n\big)\, dQ(u+) \\ &\quad + \frac{C_n}{a(n)} Q((r+1)/n+) \\ &\quad + \frac{nC_n}{a(n)}\int_{(r+1)/n}^{1-U_{n-r,n}} \big(G_n^{(2)}(u) - r/n\big)\, dQ(1-u) \\ &\quad + \frac{C_n}{a(n)} Q(1-(r+1)/n) \\ &=: \overline{M}(r,n) + R_1(r,n) + s_1(r,n) + R_2(r,n) + s_2(r,n), \qquad (2.19) \end{aligned}$$

almost surely.

LEMMA 2.7. *If* (2.3) *holds then for* $j = 1,2$ *and all* $\varepsilon > 0$,

$$\lim_{r\to\infty}\limsup_{n_1\to\infty} P\left\{|R_j(r,n_1)| \geq \varepsilon\right\} = 0.$$

Proof. Notice that on the event F_n in (1.2) we have

$$|R_j(r,n)| = \left|\int_{r+1}^{Z_{r+1,n}^{(j)}} n\big(G_n^{(j)}(u/n) - r/n\big)\, dC_n\psi_j(n,u)\right|, \qquad j = 1,2.$$

For u in the closed interval formed by $Z_{r+1,n}^{(j)}$ and $r+1$ we have

$$|G_n^{(j)}(u/n) - u/n| \leq |G_n^{(j)}((r+1)/n) - r/n|, \qquad j = 1,2.$$

Thus, on F_n, for both of the cases $j = 1$ and $j = 2$,

$$\begin{aligned} |R_j(r,n)| &\leq n|G_n^{(j)}((r+1)/n) - r/n| \\ &\qquad \times\left\{C_n|\psi_j(n, Z_{r+1,n}^{(j)})| + C_n|\psi_j(n,r+1)|\right\}. \end{aligned}$$

Let c_r and $\bar{c}_r$ be continuity points of ψ_1 satisfying $r \leq \bar{c}_r \leq r+1 \leq c_r \leq r+2$. Then for all n_1,

$$C_{n_1}\psi_j(n_1,\bar{c}_r) \leq C_{n_1}\psi_j(n_1,r+1) \leq C_{n_1}\psi_j(n_1,c_r)$$

so that

$$\psi_j(r) \le \psi_j(\bar{c}_r) \le \liminf_{n_1 \to \infty} C_{n_1} \psi_j(n_1, r+1)$$
$$\le \limsup_{n_1 \to \infty} C_{n_1} \psi_j(n_1, r+1) \le \psi_j(c_r) \le \psi_j(r+2). \tag{2.20}$$

In view of (2.17) this implies that

$$\lim_{r \to \infty} \limsup_{n_1 \to \infty} r^{1/2} C_{n_1} |\psi_j(n_1, r+1)| = 0. \tag{2.21}$$

Observe that by Chebyshev's inequality

$$\frac{n}{r^{1/2}} |G_n^{(j)}((r+1)/n) - r/n| = O_p(1) \qquad \text{as } n \to \infty \tag{2.22}$$

uniformly in $r \ge 1$. From (2.21) and (2.22) we obtain

$$\lim_{r \to \infty} \limsup_{n_1 \to \infty} P\left\{ n_1 \left| G_{n_1}^{(j)}((r+1)/n_1) - r/n_1 \right| C_{n_1} |\psi_j(n_1, r+1)| \ge \varepsilon \right\} = 0$$

for any $\varepsilon > 0$. Observing that

$$r^{1/2} \psi_j(S_{r+1}^{(j)}) \to_p 0 \qquad \text{as } r \to \infty$$

by (2.17) and combining this with (2.6) and (2.22), we get

$$\lim_{r \to \infty} \limsup_{n_1 \to \infty} P\left\{ n_1 \left| G_{n_1}^{(j)}((r+1)/n_1) - r/n_1 \right| C_{n_1} \left| \psi_j\left(n_1, Z_{r+1, n_1}^{(j)}\right) \right| \ge \varepsilon \right\} = 0$$

for any $\varepsilon > 0$. Since $P\{F_{n_1}\} \to 1$ as $n_1 \to \infty$, this completes the proof of the lemma. □

The behavior of $s_j(r, n)$ is easily obtained by arguments of the kind as applied in connection with inequality (2.20):

LEMMA 2.8. *If* (2.3) *holds then*

$$\lim_{r \to \infty} \limsup_{n_1 \to \infty} |s_j(r, n_1)| = 0, \qquad j = 1, 2.$$

According to Lemmas 2.7 and 2.8, the last four summands in the representation (2.19) all satisfy (2.18). Thus it remains to deal with $\overline{M}(r, n)$. For this we need the following result which is an easy consequence of Lemma 2.4.

LEMMA 2.9. *For any quantile function Q and for any $\nu > 0$,*

$$\limsup_{s,t\downarrow 0}\left(\sigma(s,1-t)\right)^{-1} \times\left\{s^{\nu}\int_s^{1/2}|Q(u)|u^{-1/2-\nu}\,du + t^{\nu}\int_{1/2}^{1-t}|Q(u)|(1-u)^{-1/2-\nu}\,du\right\} < \infty.$$

LEMMA 2.10. *Under the conditions of Proposition 2.2 for each fixed integer $r \geq 1$ and all n_1 sufficiently large we have*

$$\overline{M}(r,n_1) = C_{n_1}\frac{\sigma((r+1)/n_1, 1-(r+1)/n_1)}{\sigma(1/n_1, 1-1/n_1)}N_{r,n_1}(0,1) + \overline{R}(r,n_1),$$

where $N_{r,n_1}(0,1)$ is a standard normal random variable for each n_1 and r, and for each $\varepsilon > 0$,

$$\lim_{r\to\infty}\limsup_{n_1\to\infty} P\left\{|\overline{R}(r,n_1)| \geq \varepsilon\right\} = 0.$$

Proof. Let $\{B_n, n \geq 1\}$ be the sequence of Brownian bridges figuring in (2.1). For fixed $r \geq 1$ and n_1 sufficiently large

$$\begin{aligned}\overline{M}(r,n_1) &= \frac{-C_{n_1}}{\sigma(1/n_1)}\int_{[(r+1)/n_1, 1-(r+1)/n_1)} B_{n_1}(u)\,dQ(u)\\ &\quad+\frac{C_{n_1}}{\sigma(1/n_1)}\\ &\quad\times\int_{((r+1)/n_1, 1-(r+1)/n_1)}\left(B_{n_1}(u) - n_1^{1/2}\left\{G_{n_1}^{(1)}(u) - u\right\}\right)dQ(u)\\ &\quad+\frac{C_{n_1}}{\sigma(1/n_1)}B_{n_1}((r+1)/n_1)\\ &\quad\times\left\{Q((r+1)/n_1+) - Q((r+1)/n_1)\right\}\\ &= C_{n_1}\frac{\sigma((r+1)/n_1)}{\sigma(1/n_1)}N_{r,n_1}(0,1) + R'(r,n_1) + R''(r,n_1),\end{aligned}$$

where $N_{r,n_1}(0,1)$ is obviously a standard normal random variable. Observing that

$$(n_1/r)^{1/2}B_{n_1}((r+1)/n_1) = O_p(1) \qquad \text{as } n_1 \to \infty$$

holds uniformly in r, we see again that arguments following inequality

(2.20) yield that for each $\varepsilon > 0$,

$$\lim_{r\to\infty} \limsup_{n_1\to\infty} P\left\{ |R''(r, n_1)| \geq \varepsilon \right\} = 0.$$

It remains to establish the same relation for the first remainder term. To do this, fix $0 < \nu < \frac{1}{4}$ and write

$$|R'(r, n_1)| \leq \Delta(n_1) n_1^{-\nu} \frac{C_{n_1}}{\sigma(1/n_1)} \int_{(r+1)/n_1}^{1-(r+1)/n_1} (u(1-u))^{1/2-\nu}\, dQ(u),$$

where $\Delta(n_1)$ is of (2.1) and hence is $O_p(1)$ as $n_1 \to \infty$. Since the sequence $\{C_{n_1}, n_1 \geq 1\}$ is bounded and

$$0 \leq \sum(r, n_1) := \sigma((r+1)/n_1)\big/\sigma(1/n_1) \leq 1, \tag{2.23}$$

we get

$$\begin{aligned}|R'(r, n_1)| \leq O_p(1) n_1^{-\nu} (\sigma((r+1)/n_1))^{-1} \bigg\{ &\int_{(r+1)/n_1}^{1/2} u^{1/2-\nu}\, dQ(u) \\ &+ \int_{1/2}^{1-(r+1)/n_1} (1-u)^{1/2-\nu}\, dQ(u) \bigg\}\end{aligned}$$

which after an integration by parts can be bounded by

$$\begin{aligned}O_p(1)(r+1)^{-\nu} \bigg\{ (\sigma((r+1)/n_1))^{-1} &\\ \times \bigg[&((r+1)/n_1)^{1/2} |Q((r+1)/n_1)| \\ &+ (1/2 - \nu)((r+1)/n_1)^{\nu} \int_{(r+1)/n_1}^{1/2} |Q(u)| u^{-1/2-\nu}\, du \\ &+ ((r+1)/n_1)^{1/2} |Q(1-(r+1)/n_1)| \\ &+ (1/2-\nu)((r+1)/n_1)^{\nu} \\ &\times \int_{1/2}^{1-(r+1)/n_1} |Q(u)| (1-u)^{-1/2-\nu}\, du \bigg] \bigg\}.\end{aligned}$$

According to Lemmas 2.4 and 2.9, the whole factor within the curly braces stays bounded as $n_1 \to \infty$ and its bound is independent of r. Since $(r+1)^{-\nu} \to 0$ as $r \to \infty$, we arrive at

$$\lim_{r\to\infty} \limsup_{n_1\to\infty} P\left\{ |R'(r, n_1)| \geq \varepsilon \right\} = 0$$

for any $\varepsilon > 0$. This completes the proof of the lemma. □

Proposition 2.2 is now an easy consequence of Lemmas 2.7, 2.8, and 2.10, and we are ready to prove the first two theorems.

Proof of Theorem 1(i). In this case $C_n \equiv 1$ and for $T_{m,k}(n)$ in (2.5), following our convention (2.2) and the reasoning that lead to (2.19), we obtain

$$\begin{aligned}
T_{m,k}(n_1) &= \left(\sigma(1/n_1)\right)^{-1} \int_{((m+1)/n_1,\,1-(k+1)/n_1)} n_1^{1/2}\left(u - G_{n_1}^{(1)}(u)\right) dQ(u) \\
&\quad + \frac{n_1}{a(n_1)} \int_{(m+1)/n_1}^{U_{m+1,n_1}} \left(G_{n_1}^{(1)}(u) - m/n_1\right) dQ(u+) \\
&\quad + Q((m+1)/n_1+)/a(n_1) \\
&\quad + \frac{n_1}{a(n_1)} \int_{(k+1)/n_1}^{1-U_{n_1-k,n_1}} \left(G_{n_1}^{(2)}(u) - k/n_1\right) dQ(1-u) \\
&\quad + Q(1-(k+1)/n_1)/a(n_1) \\
&=: \tilde{M}_{m,k}(n_1) + \tilde{R}_m^{(1)}(n_1) + \tilde{s}_m^{(1)}(n_1) + \tilde{R}_k^{(2)}(n_1) + \tilde{s}_k^{(2)}(n_1).
\end{aligned}$$

From (1.3) and the assumption that $\psi_1 = \psi_2 \equiv 0$ we immediately get that

$$\tilde{s}_m^{(1)}(n_1) \to 0 \qquad \text{and} \qquad \tilde{s}_k^{(2)}(n_1) \to 0 \qquad \text{as } n_1 \to \infty.$$

Notice that on the event F_{n_1} of (1.2) we have for $j = 1,2$ and $h = m, k$,

$$\left|\tilde{R}_h^{(j)}(n_1)\right| = \left|\int_{h+1}^{Z_{h+1}^{(j)}} n_1\left(G_{n_1}^{(j)}(u/n_1) - h/n_1\right) d\psi_j(n_1, u)\right|$$

and, as in the proof of Lemma 2.7,

$$\begin{aligned}
\left|\tilde{R}_h^{(j)}(n_1)\right| &\le \left|n_1\left(G_{n_1}^{(j)}((h+1)/n_1) - h/n_1\right)\right| \\
&\qquad \times \left\{\left|\psi_j(n_1, Z_{h+1}^{(j)})\right| + \left|\psi_j(n_1, h+1)\right|\right\}.
\end{aligned}$$

The first factor on the right side is $O_p(1)$ as $n_1 \to \infty$, as can be seen by the Chebyshev inequality, and the second factor is $o_p(1)$ as $n_1 \to \infty$ as a consequence of (1.3) with $\psi_j \equiv 0$ and (2.6) of Lemma 2.1. Therefore, since $P\{F_{n_1}\} \to 1$ as $n_1 \to \infty$,

$$\tilde{R}_h^{(j)}(n_1) \to_p 0 \qquad \text{as } n_1 \to \infty,\ j = 1,2;\ h = m, k.$$

Furthermore, similarly as in the proof of Lemma 2.10,

$$
\begin{aligned}
&\tilde{M}_{m,k}(n_1)\\
&\quad = N_{n_1}(0,1) + \big(\sigma(1/n_1)\big)^{-1}\\
&\qquad \times \int_{((m+1)/n_1,\,1-(k+1)/n_1)} \Big\{ B_{n_1}(u) - n_1^{1/2}\big(G_{n_1}^{(1)}(u) - u\big)\Big\}\, dQ(u)\\
&\qquad + \big(\sigma(1/n_1)\big)^{-1} \int_{1/n_1}^{(m+1)/n_1} B_{n_1}(u)\, dQ(u)\\
&\qquad + \big(\sigma(1/n_1)\big)^{-1} \int_{1-(k+1)/n_1}^{1-1/n_1} B_{n_1}(u)\, dQ(u)\\
&\qquad + \big(\sigma(1/n_1)\big)^{-1} B_{n_1}\big((m+1)/n_1\big)\\
&\qquad\quad \times \big\{Q\big((m+1)/n_1+\big) - Q\big((m+1)/n_1\big)\big\}\\
&\quad =: N_{n_1}(0,1) + \overline{R}_{m,k}^{(1)}(n_1) + \overline{R}_{m}^{(2)}(n_1) + \overline{R}_{k}^{(3)}(n_1) + \overline{R}_{m}^{(4)}(n_1),
\end{aligned}
$$

where $N_{n_1}(0,1)$ is a standard normal random variable for each n_1. Now

$$
\begin{aligned}
\overline{R}_m^{(4)}(n_1) &= n_1^{1/2} B_{n_1}\big((m+1)/n_1\big)\\
&\qquad \times \left\{ \frac{Q\big((m+1)/n_1+\big)}{a(n_1)} - \frac{Q\big((m+1)/n_1\big)}{a(n_1)} \right\},
\end{aligned}
$$

where, obviously,

$$
n_1^{1/2} B_{n_1}\big((m+1)/n_1\big) = O_p(1) \qquad \text{as } n_1 \to \infty,
$$

and the other factor goes to zero as $n_1 \to \infty$ as a consequence of (1.3) with $\psi_1 \equiv 0$. This proves that

$$
\overline{R}_m^{(4)}(n_1) \to_p 0 \qquad \text{as } n_1 \to \infty.
$$

Since

$$
\begin{aligned}
E\big|\overline{R}_m^{(2)}(n_1)\big| &\le \big(\sigma(1/n_1)\big)^{-1} \int_{1/n_1}^{(m+1)/n_1} u^{1/2}\, dQ(u)\\
&\le (m+1)^{1/2} \big\{Q\big((m+1)/n_1\big) - Q(1/n_1)\big\} \big/ a(n_1),
\end{aligned}
$$

condition (1.3) with $\psi_1 \equiv 0$ again implies that $E|\overline{R}_m^{(2)}(n_1)| \to 0$ as $n_1 \to \infty$, which by the Markov inequality gives

$$
\overline{R}_m^{(2)}(n_1) \to_p 0 \qquad \text{as } n_1 \to \infty.
$$

Exactly the same reasoning (with $\psi_2 \equiv 0$) yields

$$\overline{R}_k^{(3)}(n_1) \to_p 0 \qquad \text{as } n_1 \to \infty.$$

In order to estimate the first remainder term, we fix $\nu \in (0, \frac{1}{4})$ and repeat the corresponding argument in the proof of Lemma 2.10 to obtain

$$\begin{aligned}
&\left|\overline{R}_{m,k}^{(1)}(n_1)\right| \\
&\quad \le O_p(1)\left\{\frac{|Q((m+1)/n_1)|}{a(n_1)} + \frac{n_1^{1/2-\nu}}{a(n_1)}\int_{(m+1)/n_1}^{1/2}|Q(u)|u^{-1/2-\nu}\,du\right. \\
&\qquad + \frac{|Q(1-(k+1)/n_1)|}{a(n_1)} \\
&\qquad \left. + \frac{n_1^{1/2-\nu}}{a(n_1)}\int_{1/2}^{1-(k+1)/n_1}|Q(u)|(1-u)^{-1/2-\nu}\,du\right\}.
\end{aligned}$$

The first and the third summands in the curly braces converge to zero on account of (1.3) with $\psi_1 = \psi_2 \equiv 0$. Concerning the second summand, first note that it converges to zero trivially as $n_1 \to \infty$ if Q is non-negative on $(0, \frac{1}{2})$. In the opposite case choose a number $K \in (m+1, \infty)$ and bound this third summand by

$$\frac{n_1^{1/2-\nu}}{a(n_1)}\int_{(m+1)/n_1}^{K/n_1}|Q(u)|u^{-1/2-\nu}\,du + \frac{n_1^{1/2-\nu}}{a(n_1)}\int_{K/n_1}^{1/2}|Q(u)|u^{-1/2-\nu}\,du.$$

Here, for large enough n_1, the first term is not greater than

$$\begin{aligned}
&\frac{n_1^{1/2-\nu}}{a(n_1)}|Q((m+1)/n_1)|\int_{(m+1)/n_1}^{K/n_1} u^{-1/2-\nu}\,du \\
&\qquad \le (1/2-\nu)^{-1}K^{1/2-\nu}\frac{|Q((m+1)/n_1)|}{a(n_1)}
\end{aligned}$$

which goes to zero by (1.3) with $\psi_1 \equiv 0$ for any fixed K as $n_1 \to \infty$, while the second term is bounded by

$$n_1^{-\nu}\int_{K/n_1}^{1/2}\frac{u^{1/2}|Q(u)|}{\sigma(u)}\,\frac{\sigma(u)}{\sigma(1/n_1)}u^{-1-\nu}\,du \le Cn_1^{-\nu}\int_{K/n_1}^{1/2}u^{-1-\nu}\,du,$$

where C is a finite constant according to Lemma 2.4. The latter bound converges to $C\nu^{-1}K^{-\nu}$ as $n_1 \to \infty$, and this can be made arbitrarily small by choosing K large enough. Thus the second summand in question converges to zero as $n_1 \to \infty$. The fourth summand approaches zero as

$n_1 \to \infty$ for similar reasons. Thus we have proved

$$\overline{R}^{(1)}_{m,k}(n_1) \to_p 0 \qquad \text{as } n_1 \to \infty,$$

and combining this with the foregoing, the proof of Theorem 1(i) is complete. □

Proof of Theorem 1(ii). We simply set $r = sl$ in the representation in (2.5), where $s \geq 2$ is an integer. According to Proposition 2.1, for any fixed $l \geq 1$ we have

$$\max_{0 \leq h \leq l} \left| V_h^{(j)}(l, n_1) - V_h^{(j)}(l) \right| \to_p 0 \qquad \text{as } n_1 \to \infty,\ j = 1,2. \quad (2.24)$$

By elementary calculation, for $j = 1,2$ and $h = m, k$,

$$\begin{aligned} V_h^{(j)} - V_h^{(j)}(l) = (-1)^{j+1} \bigg\{ & \int_{S_{l+1}^{(j)}}^{\infty} \big(u - N_j(u)\big)\, d\psi_j(u) \\ & + \int_{l+1}^{S_{l+1}^{(j)}} (u - (l+1))\, d\psi_j(u) \\ & + \psi_j\big(S_{l+1}^{(j)}\big) \bigg\}, \end{aligned}$$

so that this difference does not depend on h. Observe also that square integrability of the ψ_j, as proved in Lemma 2.5, implies that the improper integrals

$$\int_{S_{l+1}^{(j)}}^{\infty} \big(u - N_j(u)\big)\, d\psi_j(u)$$

exist with probability one. Combining this fact, the strong law of large numbers and Lemma 2.6, we obtain

$$\max_{0 \leq h \leq l} \left| V_h^{(j)}(l) - V_h^{(j)} \right| \to_p 0 \text{ as } l \to \infty,\ j = 1,2. \quad (2.25)$$

Next observe that the random variables $\Delta_j(l, sl, n)$ in (2.5) have the same structure as $V_m^{(j)}(l, n)$, namely in this notation $\Delta_j(l, sl, n) = V_l^{(j)}(sl, n)$, $j = 1,2$. So we can apply Proposition 2.1 to them as well. Thus we obtain that for any fixed $l \geq 1$,

$$\begin{aligned} \Delta_j(l, sl, n_1) &\to_p \Delta_j(s, l) \\ &:= (-1)^{j+1} \bigg\{ \int_{S_{l+1}^{(j)}}^{S_{sl+1}^{(j)}} \big(u - N_j(u)\big)\, d\psi_j(u) \\ &\qquad + \int_{l+1}^{S_{l+1}^{(j)}} (u - (l+1))\, d\psi_j(u) + \psi_j\big(S_{l+1}^{(j)}\big) \\ &\qquad + \int_{S_{sl+1}^{(j)}}^{sl+1} (u - (sl+1))\, d\psi_j(u) - \psi_j\big(S_{sl+1}^{(j)}\big) \bigg\} \quad (2.26) \end{aligned}$$

as $n_1 \to \infty$. Also from the existence of the integrals

$$\int_1^\infty \big(u - N_j(u)\big)\, d\psi_j(u), \qquad j = 1,2,$$

and Lemma 2.6 we see that for each fixed $s \geq 2$,

$$\Delta_j(s, l) \to_p 0 \qquad \text{as } l \to \infty, \qquad j = 1,2. \tag{2.27}$$

For any random variable Z we set

$$\rho(Z) = \inf\{\varepsilon > 0\colon P\{|Z| \geq \varepsilon\} < \varepsilon\}.$$

It is well known that for a sequence of random variables $Z_1, Z_2, \ldots$ we have $Z_n \to_p Z$ as $n \to \infty$ if and only if $\rho(Z_n - Z) \to 0$ as $n \to \infty$.

Noting that (2.18) implies that for each fixed $s \geq 2$,

$$\lim_{l \to \infty} \limsup_{n_1 \to \infty} \rho\big(R(sl, n_1)\big) = 0, \tag{2.28}$$

relations (2.25), (2.27), and (2.28) and the square integrability of the ψ_j functions allow us to construct a strictly increasing sequence $\{\bar{l}_s, s \geq 1\}$ of positive integers such that for $s \geq 2$,

$$\rho\Big(\max_{0 \leq h \leq \bar{l}_s} \big|V_h^{(j)}(\bar{l}_s) - V_h^{(j)}\big|\Big) \leq s^{-1}, \qquad \rho\big(\Delta_j(s, \bar{l}_s)\big) \leq s^{-1}, \quad j = 1,2,$$

$$\limsup_{n_1 \to \infty} \rho\big(R(s\bar{l}_s, n_1)\big) \leq s^{-1} \qquad \text{and} \qquad s\bar{l}_s \psi_j^2(\bar{l}_s) \leq s^{-1}, \quad j = 1,2.$$

Using now (2.24) and (2.26) and (1.3) in conjunction with the last seven inequalities, we can inductively construct two sequences $\{s_{n_1}\}$ and $\{l_{n_1}\}$ of positive integers and a sequence $\{\varepsilon_{n_1}\}$ of positive numbers such that $s_{n_1} \to \infty$, $l_{n_1} \to \infty$, and $\varepsilon_{n_1} \to 0$ as $n_1 \to \infty$ and for each n_1 in $\{n_1\}$ the following twelve inequalities hold simultaneously:

$$\rho\Big(\max_{0 \leq h \leq l_{n_1}} \big|V_h^{(j)}(l_{n_1}, n_1) - V_h^{(j)}(l_{n_1})\big|\Big) \leq \varepsilon_{n_1},$$

$$\rho\Big(\max_{0 \leq h \leq l_{n_1}} \big|V_h^{(j)}(l_{n_1}) - V_h^{(j)}\big|\Big) \leq \varepsilon_{n_1},$$

$$\rho\big(\Delta_j(l_{n_1}, s_{n_1} l_{n_1}, n_1) - \Delta_j(s_{n_1}, l_{n_1})\big) \leq \varepsilon_{n_1},$$

$$\rho\big(\Delta_j(s_{n_1}, l_{n_1})\big) \leq \varepsilon_{n_1}, \qquad j = 1,2,$$

$$\rho\big(R(s_{n_1}, l_{n_1}, n_1)\big) \leq \varepsilon_{n_1},$$

$$s_{n_1} l_{n_1} \psi_j^2(n_1, l_{n_1}) \leq \varepsilon_{n_1}, \qquad j = 1,2,$$

$$s_{n_1} l_{n_1} / n_1 \leq \varepsilon_{n_1}.$$

Hence, it follows by the triangle inequality for ρ that for all n_1

$$\rho\left(\max_{0\le h\le l_{n_1}}\left|V_h^{(j)}\left(l_{n_1}, n_1\right) - V_h^{(j)}\right|\right) \le 2\varepsilon_{n_1}$$

and

$$\rho\left(\Delta_j\left(l_{n_1}, s_{n_1}l_{n_1}, n_1\right)\right) \le 2\varepsilon_{n_1},$$

$j = 1, 2$. This means, on account of (2.5) (recalling that $C_{n_1} \equiv 1$ in the present case (ii)), that we have constructed two sequences $\{l_{n_1}\}$ and $\{r_{n_1}\} = \{s_{n_1}l_{n_1}\}$ such that (1.5) and (1.6) hold and (1.7) and (1.8) are also satisfied for any fixed choice of $m \ge 0$ and $k \ge 0$. (We emphasize that the constructed sequences $\{l_{n_1}\}$ and $\{r_{n_1}\}$ are independent of fixed choices of $m \ge 0$ and $k \ge 0$.)

Since $R(r_{n_1}, n_1) \to_p 0$ as $n_1 \to \infty$, it follows from Proposition 2.2 in the present case (ii) that

$$M\left(r_{n_1}, n_1\right) = \sum\left(r_{n_1}, n_1\right)N_{n_1}(0,1) + o_p(1) \qquad \text{as } n_1 \to \infty,$$

where $N_{n_1}(0,1)$ is a standard normal random variable for each n_1 and $\Sigma(r_{n_1}, n_1)$, defined also in (2.23), is already the quantity figuring in the formulation of the theorem. Consequently, the statement in (1.9) follows trivially, and thus we proved all the claimed convergence statements.

Furthermore, $V_m^{(1)}$ and $V_k^{(2)}$ are independent by construction. This is of course not enough for the claim above (1.10), but (1.5), (1.6), and Satz 4 of Rossberg [31] imply that the three sequences of random variables on the left sides of (1.8) and (1.9) are asymptotically independent. Therefore, the independence claim is true in its full generality. That $V_m^{(1)}$ is non-degenerate if $\psi_1 \not\equiv 0$ and $V_k^{(2)}$ is non-degenerate if $\psi_2 \not\equiv 0$ is included and proved in Theorem 4.

It remains to verify (1.10) when $\sigma > 0$. For $0 < t < s < \frac{1}{2}$ we have the identity

$$\begin{aligned}\sigma^2(t) - \sigma^2(s) = &\int_t^s\int_t^s (u \wedge v - uv)\, dQ(u)\, dQ(v)\\ &+ \int_{1-s}^{1-t}\int_{1-s}^{1-t}(u \wedge v - uv)\, dQ(u)\, dQ(v)\\ &+ 2\int_s^{1-s}\int_t^s (u \wedge v - uv)\, dQ(u)\, dQ(v)\\ &+ 2\int_s^{1-s}\int_{1-s}^{1-t}(u \wedge v - uv)\, dQ(u)\, dQ(v)\\ &+ 2\int_{1-s}^{1-t}\int_t^s (u \wedge v - uv)\, dQ(u)\, dQ(v), \qquad (2.29)\end{aligned}$$

from which one readily obtains the estimate

$$
\begin{aligned}
1 &\le \frac{\sigma^2(t)}{\sigma^2(s)} \\
&\le 1 + 3\left\{ s\frac{Q^2(s) + Q^2(t) + Q^2(1-s) + Q^2(1-t)}{\sigma^2(s)} \right. \\
&\qquad + s^{1/2}\frac{\left(Q^2(s) + Q^2(t)\right)^{1/2} + \left(Q^2(1-s) + Q^2(1-t)\right)^{1/2}}{\sigma(s)} \\
&\qquad \left. + s^2\frac{|Q(s)| + |Q(t)|}{\sigma(s)}\,\frac{|Q(1-s)| + |Q(1-t)|}{\sigma(s)} \right\}. \qquad (2.30)
\end{aligned}
$$

Substitute $t = (l_{n_2} + 1)/n_2$ and $s = (r_{n_2} + 1)/n_2$ into (2.30) to obtain a bound on the ratio $\sigma((l_{n_2} + 1)/n_2)/\sigma((r_{n_2} + 1)/n_2)$. Then $sQ^2(s)/\sigma^2(s)$ becomes

$$\frac{r_{n_2} + 1}{n_2}\,\frac{Q^2\big((r_{n_2} + 1)/n_2\big)}{\sigma^2\big((r_{n_2} + 1)/n_2\big)} \le \frac{\sigma^2(1/n_2)}{\sigma^2\big((r_{n_2} + 1)/n_2\big)}\big(r_{n_2} + 1\big)\psi_1^2\big(n_2, r_{n_2}\big),$$

where the right side converges to zero as $n_2 \to \infty$, since

$$\sigma^2(1/n_2)\big/\sigma^2\big((r_{n_2} + 1)/n_2\big) \to 1/\sigma < \infty \qquad \text{as } n_2 \to \infty$$

and

$$r_{n_2}\psi_1^2\big(n_2, r_{n_2}\big) \le s_{n_2} l_{n_2}\psi_1^2\big(n_2, l_{n_2}\big) \to 0 \qquad \text{as } n_2 \to \infty.$$

Furthermore, $sQ^2(t)/\sigma^2(s)$ becomes

$$\frac{r_{n_2} + 1}{n_2}\,\frac{Q^2\big((l_{n_2} + 1)/n_2\big)}{\sigma^2\big((r_{n_2} + 1)/n_2\big)} \le \frac{\sigma^2(1/n_2)}{\sigma^2\big((r_{n_2} + 1)/n_2\big)}\big(r_{n_2} + 1\big)\psi_1^2\big(n_2, l_{n_2}\big),$$

and this goes to zero as $n_2 \to \infty$ by the same reasons. All the other terms in the bound on $\sigma((l_{n_2} + 1)/n_2)/\sigma((r_{n_2} + 1)/n_2)$ obtained from (2.30) can be shown to converge to zero by similar arguments. This completes the proof of Theorem 1(ii). □

Proof of Theorem 2. Now we set $r = l$ in the representation (2.5) so that $T_{m,k}(n) = V_m^{(1)}(l, n) + M(l, n) + V_k^{(2)}(l, n)$. Exactly the same arguments as in the proof of part (ii) of Theorem 1 lead to the construction of a sequence $\{l_{n_1}\}$ of positive integers such that (1.13) and (1.15) hold as

$n_1 \to \infty$ and for the middle term in (1.14)

$$M(l_{n_1}, n_1) = C_{n_1} \Sigma(l_{n_1}, n_1) N_{n_1}(0,1) + o_p(1) \qquad \text{as } n_1 \to \infty,$$

where $N_{n_1}(0,1)$ is a standard normal random variable for each $n_1 \geq 1$ and $\Sigma(\cdot,\cdot)$ is defined in (2.23). Since

$$C_{n_1} = a(n_1)/A_{n_1} \to 0 \qquad \text{as } n_1 \to \infty$$

by assumption (1.12), we get

$$M(l_{n_1}, n_1) \to_p 0 \qquad \text{as } n_1 \to \infty.$$

This proves (1.14). The statement in (1.16) follows by construction. The nondegeneracy statement follows again from Theorem 4. Finally, observe that for $1 < s < \infty$ and all large n_1, assuming that Q is negative near zero,

$$\begin{aligned} \frac{a(n_1)}{A_{n_1}} |\psi_1(n_1, s)| &= o(1)(1/n_1)^{1/2} |Q(s/n_1 +)|/\sigma(1/n_1) \\ &\leq o(1)(1/n_1)^{1/2} |Q(1/n_1 +)|/\sigma(1/n_1) = o(1) \quad (2.31) \end{aligned}$$

as $n_1 \to \infty$ by Lemma 2.4, so that $\psi_1(s) = 0$ for $1 < s < \infty$. If Q is never negative this is trivial. That $\psi(1) = 0$ follows by right-continuity. Analogously, $\psi_2(s) = 0$ if $1 \leq s < \infty$. The theorem is completely proved. □

Proof of Theorem 3(i). For $T > 1$, define

$$\begin{aligned} V(T) &= \int_1^T s\, d\psi(s) - \int_0^T N(s)\, d\psi(s) \\ &= T\psi(T) - \psi(1) - \int_1^T \psi(s)\, ds - \int_0^T N(s)\, d\psi(s) \end{aligned}$$

and notice that

$$\begin{aligned} V(T) &= \sum_{k=1}^{N(T)} \psi(S_k) - \psi(T)N(T) + T\psi(T) - \psi(1) - \int_1^T \psi(s)\, ds \\ &=: \overline{V}(T) - \psi(T)(N(T) - T), \end{aligned}$$

where the empty sum is understood as zero. Since $(N(T) - T)/T^{1/2} \to_D N(0,1)$ and the square integrability of ψ implies that $T^{1/2}\psi(T) \to 0$ as $T \to \infty$, we have $\psi(T)(N(T) - T) \to_p 0$ as $T \to \infty$. Therefore it is enough to show that for any real t,

$$\bar{\phi}_T(t) := Ee^{it\bar{V}(T)} \to Ee^{it(V_0 - \psi(1))} = \phi_0(t)e^{-it\psi(1)} \qquad \text{as } T \to \infty. \quad (2.32)$$

It is well known that the conditional distribution of $S_1, \dots, S_k$ given $N(T) = k$ is the distribution of the order statistics of k independent uniform $(0, T)$ random variables. Hence

$$\begin{aligned}
\bar{\phi}_T(t) &= \sum_{k=0}^{\infty} \left(\frac{1}{T} \int_0^T e^{it\psi(u)}\, du \right)^k \frac{T^k}{k!} e^{-T} \exp\left(-it\left(\int_1^T \psi(s)\, ds + \psi(1) \right)\right) \\
&= \exp\left(\int_0^T (e^{it\psi(u)} - 1)\, du - it \int_1^T \psi(u)\, du - it\psi(1) \right) \\
&= \exp\Bigg(\int_1^T \{ e^{it\psi(u)} - 1 - it\psi(u) \}\, du \\
&\qquad + \int_0^1 \left\{ e^{it\psi(u)} - 1 - \frac{it\psi(u)}{1 + \psi^2(u)} \right\} du \\
&\qquad + it \int_0^1 \frac{\psi(u)}{1 + \psi^2(u)}\, du - it\psi(1) \Bigg)
\end{aligned}$$

which, as $T \to \infty$, converges to

$$\begin{aligned}
\exp\Bigg(\int_1^\infty \{ e^{it\psi(u)} - 1 - it\psi(u) \}\, du + \int_0^1 \left\{ e^{it\psi(u)} - 1 - \frac{it\psi(u)}{1 + \psi^2(u)} \right\} du \\
+ it \int_0^1 \frac{\psi(u)}{1 + \psi^2(u)}\, du - it\psi(1) \Bigg)
\end{aligned}$$

A simple rearrangement shows that this limit is $\phi_0(t)\exp(-it\psi(1))$. Noting that all the integrals appearing in the formulation of the theorem and in the proof above are finite by the square integrability condition on ψ, the proof of (2.32) is complete. We recognize ϕ_0 as the characteristic function of an infinitely divisible law (cf. Gnedenko and Kolmogorov [14, p. 84]) and the uniqueness of the Lévy measures and the constants of these laws implies that if $\psi \not\equiv 0$ then V_0 is non-degenerate. This proves part (i) of the theorem and part (ii) follows from part (i) by simple considerations. □

Proof of Theorem 4. We only have to establish the formula for the characteristic function. Using the notation in the formulation and the proof of part (i) of Theorem 3, consider

$$\begin{aligned} V_h = V_h(S_{h+1}) &= \int_{S_{h+1}}^{\infty} (u - N(u))\, d\psi(u) \\ &\qquad + \int_1^{S_{h+1}} u\, d\psi(u) - h\psi(S_{h+1}) + \int_1^{h+1} \psi(u)\, du + \psi(1) \\ &= \int_0^{\infty} (s + S_{h+1} - N(s + S_{h+1}))\, d\psi(s + S_{h+1}) \\ &\quad + \int_{1-S_{h+1}}^{0} (s + S_{h+1})\, d\psi(s + S_{h+1}) \\ &\quad - h\psi(S_{h+1}) + \int_1^{h+1} \psi(u)\, du + \psi(1) \end{aligned}$$

as a function of S_{h+1}. Introduce the new Poisson process

$$N'(s) = \sum_{k=0}^{\infty} I(S'_k < s), \qquad s \geq 0,$$

where $S'_k = Y_{h+1} + \cdots + Y_{h+1+k}$, $k \geq 0$, and note that

$$N(u) = N'(u - S_{h+1}) + \sum_{j=1}^{h+1} I(S_j < u), \qquad u \geq 0.$$

So, conditional on $S_{h+1} = x$ we obtain by elementary computations that for any $x > 0$,

$$V_h(x) = \int_{S'_1}^{\infty} (s - N'(s))\, d\psi_x(s) + \int_1^{S'_1} s\, d\psi_x(s) + \psi_x(1) + c_h(x),$$

where $\psi_x(s) = \psi(x + s)$, $s \geq 0$, and

$$c_h(x) = \psi(x) + \int_{x+1}^{h+1} \psi(u)\, du.$$

Using the fact that (1.17) holds under the conditions of the theorem, the formula follows by the fact that $V_m^{(1)}$, $N(0, \sigma^2)$, and $V_k^{(2)}$ are independent and an easy application of the first part of Theorem 3.

Now let $h = m$ if $j = 1$ and $h = k$ if $j = 2$, and assume that $V_h^{(j)}$ is a degenerate random variable. Then, dropping the subscripts j and superscripts (j) everywhere for convenience, so is

$$\int_{S_{h+1}}^{\infty} (u - N(u))\, d\psi(u) + \int_1^{S_{h+1}} u\, d\psi(u) - h\psi(S_{h+1}),$$

which by integrating by parts and substitution can be written as

$$\zeta - \psi(1) := \int_0^\infty \{(u + S_{h+1}) - N(u + S_{h+1}) - (S_{h+1} - N(S_{h+1}))\}\, d\psi(u + S_{h+1}) - \int_1^{S_{h+1}} \psi(u)\, du - \psi(1).$$

Thus ζ is degenerate at some point c and hence independent of S_{h+1}. Therefore, for any $x > 0$, the conditional distribution of ζ given $S_{h+1} = x$ is also concentrated at c. On the other hand, since the process $\{u - N(u), u \geq 0\}$ has stationary and independent increments and since S_{h+1} is a stopping time for this process, and hence the process

$$\{(u + S_{h+1}) - N(u + S_{h+1}) - (S_{h+1} - N(S_{h+1})), u \geq 0\}$$

and S_{h+1} are independent, this conditional distribution is the same as the (unconditional) distribution of

$$\int_0^\infty \{(x + u) - N(x + u) - (x - N(x))\}\, d\psi_x(u) - \int_1^x \psi(u)\, du.$$

The latter is equal in distribution to

$$\eta(x) := \int_0^\infty (u - N(u))\, d\psi_x(u) - \int_1^x \psi(u)\, du.$$

Consequently $P\{\eta(x) = c\} = 1$ and hence

$$0 = \operatorname{Var}\left\{\int_0^\infty (u - N(u))\, d\psi_x(u)\right\} = \int_0^\infty \int_0^\infty (u \wedge v)\, d\psi_x(u)\, d\psi_x(v).$$

This can only happen if $\psi_x \equiv 0$. Since this holds for all $x > 0$, we must have $\psi = \psi_j \equiv 0$. This completely proves the theorem. □

Proof of Theorem 5. We distinguish two cases: Either

$$\limsup_{n_1 \to \infty} |\psi_1(n_1, s)| < \infty, \qquad 0 < s < \infty \tag{2.33}$$

and

$$\limsup_{n_1 \to \infty} |\psi_2(n_1, s)| < \infty, \qquad 0 < s < \infty, \tag{2.34}$$

or (2.33) or (2.34) fails for some $s > 0$.

First we consider the first case when (2.33) and (2.34) both hold. By the Helly–Bray theorem there exists a subsequence $\{n_2\}$ of $\{n_1\}$ such that

$$\lim_{n_2\to\infty} \psi_j(n_2, s) = \psi_j(s), \qquad 0 < s < \infty,\ j = 1,2, \tag{2.35}$$

where ψ_1 and ψ_2 are non-decreasing, non-positive, right-continuous functions and the relations in (2.35) hold at all continuity points of the respective ψ functions. By Theorem 1 there exists a further subsequence $\{n_3\}$ of $\{n_2\}$ such that

$$\frac{1}{a(n_3)}\left\{\sum_{j=m+1}^{n_3-k} X_{j,n_3} - n_3\mu_{m,k}(n_3)\right\}$$

converges in distribution to a non-degenerate random variable. Therefore by the convergence of types theorem [14, p. 40–42] we have

$$a(n_3)/A_{n_3} \to \delta > 0 \qquad \text{as } n_3 \to \infty$$

and

$$\tilde{B}_{n_3} = \left(n_3\mu_{m,k}(n_3) - B_{n_3}\right)/A_{n_3} \to K \qquad \text{as } n_3 \to \infty, \tag{2.36}$$

where K is a finite constant. Hence (1.19) and (1.20) along $\{n_3\}$ follow with ψ^* functions being constant multiples of the respective ψ functions in (2.35). The last claim of the theorem also follows in this case with $\sigma \geq 0$ by Theorems 1 or 4.

Now we turn to the second case. More specifically we assume that for some $s > 0$ and some subsequence $\{n_2\}$ of $\{n_1\}$ we have

$$\lim_{n_2\to\infty} \psi_1(n_2, s) = -\infty. \tag{2.37}$$

Note right away that by Lemma 2.4, using the inequality in (2.31) (but not the equalities), for this s we necessarily have

$$0 < s < 1. \tag{2.38}$$

We write

$$\frac{1}{A_n}\left\{\sum_{j=m+1}^{n-k} X_{j,n} - B_n\right\} = \frac{a(n)}{A_n} T_{m,k}(n) + \tilde{B}_n,$$

where $T_{m,k}(n)$ is as in (2.5) with $C_n \equiv 1$ and $\tilde{B}_n$ is defined in (2.36). Using the decomposition (2.3) in S. Csörgő, Haeusler, and Mason [4], one can write

$$T_{m,k}(n) = \left\{ \frac{Q(U_{m+1,n})}{a(n)} + \frac{n}{a(n)} \int_{(m+1)/n}^{U_{m+1,n}} \left(G_n(s) - \frac{m+1}{n} \right) dQ(s) \right\}$$
$$+ \frac{n}{a(n)} \int_{(m+1)/n}^{1-(k+1)/n} (s - G_n(s))\, dQ(s)$$
$$+ \left\{ \frac{Q(U_{n-k,n})}{a(n)} \right.$$
$$\left. + \frac{n}{a(n)} \int_{U_{n-k-1,n}}^{1-(k+1)/n} \left(G_n(s) - \frac{n-k-1}{n} \right) dQ(s) \right\}$$
$$=: R_n^{(1)} + W_n + R_n^{(2)},$$

where

$$G_n(s) = n^{-1} \sum_{j=1}^{n} I(U_{j,n} \le s)$$

is the right-continuous version of our so far used $G_n^{(1)}$. We need three lemmas.

LEMMA 2.11. *For* $j = 1, 2$,

$$\lim_{M\to\infty} \liminf_{n\to\infty} P\left\{ |R_n^{(j)}| < M \right\} > 0. \tag{2.39}$$

Proof. We consider only the case $j = 1$, the other is similar. Observe that

$$|R_n^{(1)}| \le n \left| G_n\left(\frac{m+1}{n} \right) - \frac{m+1}{n} \right| \frac{|Q(U_{m+1,n}) - Q((m+1)/n)|}{a(n)}$$
$$+ \frac{|Q(U_{m+1,n})|}{a(n)}.$$

We see that on the event $A(n) = \{(m+1)/n < U_{m+1,n} < 2(m+1)/n\}$,

$$|R_n^{(1)}| \le 3\left\{ n\left(G_n\left(\frac{m+1}{n} \right) + \frac{m+1}{n} \right) + 1 \right\}$$
$$\times \left\{ \left(\left| Q\left(\frac{m+1}{n} \right) \right| + \left| Q\left(\frac{2(m+1)}{n} \right) \right| \right) \Big/ a(n) \right\}.$$

Here the first factor is obviously $O_p(1)$, while by Lemma 2.4 the second one is bounded for all large enough n. Since

$$\lim_{n\to\infty} P\{A(n)\} = P\{m+1 < S_{m+1} < 2(m+1)\} > 0,$$

this proves (2.39). □

The next lemma follows from a special case of Satz 4 of Rossberg [31] by an elementary method similar to the one in the proof of Theorem 2 of Mason [27]. The simple details are omitted.

LEMMA 2.12. *The two sequences of random variables* $|R_n^{(1)}|$ *and* $|R_n^{(2)}|$ *are asymptotically independent.*

Now Lemmas 2.11, 2.12, and Lemma 2.10 in S. Csörgő, Haeusler, and Mason [4] allow us to argue exactly as in the proof of Lemma 2.11 of that same paper to obtain the following.

LEMMA 2.13. *Whenever there exists a subsequence* $\{n_1\}$ *of* $\{n\}$ *with accompanying normalizing and centering constants* $A_{n_1} > 0$ *and* B_{n_1} *such that the sequence in* (1.18) *is stochastically bounded, then both sequences*

$$a(n_1)R_{n_1}^{(j)}/\left(A_{n_1} \vee a(n_1)\right), \qquad j = 1,2,$$

are stochastically bounded.

Coming back to the proof of Theorem 5, to relation (2.37), we claim now that

$$\limsup_{n_2\to\infty} a(n_2)|\psi_1(n_2,s)|/A_{n_2} < \infty, \qquad 0 < s < \infty, \tag{2.40}$$

and

$$\lim_{n_2\to\infty} a(n_2)/A_{n_2} = 0. \tag{2.41}$$

On the event $B(n_2) = \{U_{m+1,n_2} < s/n_2\}$, where $s \in (0,1)$ is as in (2.37), we have

$$\begin{aligned} a(n_2)\left|R_{n_2}^{(1)}\right|/\left(A_{n_2} \vee a(n_2)\right) &\ge |Q(s/n_2+)|/\left(A_{n_2} \vee a(n_2)\right) \\ &= a(n_2)|\psi_1(n_2,s)|/\left(A_{n_2} \vee a(n_2)\right) \end{aligned} \tag{2.42}$$

because the integral term in $R_n^{(1)}$ is non-positive for large enough n and $Q(s/n_2)$ is non-positive for large enough n_2 (otherwise (2.37) could not happen). Knowing that $P\{B(n_2)\} \to P\{S_{m+1} < s\} > 0$, as $n_2 \to \infty$, and that the left side of (2.42) is stochastically bounded by Lemma 2.13, and taking into account (2.37), we indeed obtain (2.40) and (2.41).

Now choose a further subsequence $\{n_3\}$ of $\{n_2\}$ by repeating the above proof if necessary to arrive at

$$\limsup_{n_3 \to \infty} a(n_3)|\psi_2(n_3, s)|/A_{n_3} < \infty, \qquad 0 < s < \infty, \tag{2.43}$$

paralleling (2.40). By a final application of the Helly–Bray theorem choose the subsequence $\{n_4\}$ of $\{n_3\}$ such that the two relations in (1.20) hold along $\{n_4\}$. Since (2.41) holds along $\{n_4\}$, Lemma 2.5 implies (1.19). This finishes the proof of the first part of the theorem in that subcase of the second case when (2.33) is not satisfied. Since we also have (2.36) along $\{n_4\}$, an application of Theorem 2 gives the last claim of the theorem in the present subcase.

Since the proof in the second subcase of the second case when (2.34) is not satisfied is entirely analogous, the theorem is completely proved. □

In the proofs of Corollaries 1, 2, 8, and 10 below we prove only the equivalence of the two convergence statements and the corresponding a-type analytic condition in the present section. The proofs that the corresponding a–b–c analytic conditions are themselves equivalent are similar to each other. Therefore, in order to save space and to concentrate first on the main issues, these analytic proofs are given together in Section 3. At one small place, however, in the proof of the last statement of Corollary 10, we shall use condition (1.42c) already in the present section.

Proof of Corollary 1. First assume (1.25). Then there exist sequences $A_n > 0$ and B_n such that, as $n \to \infty$,

$$A_n^{-1}\{S_n(m,k) - B_n\} \to_D N(\mu, \sigma^2) \quad \text{for some } \mu \in (-\infty, \infty) \text{ and } \sigma > 0. \tag{2.44}$$

We claim that as $n \to \infty$,

$$H(\lambda/n)/a(n) \to 0 \qquad \text{for all } 0 < \lambda < \infty. \tag{2.45}$$

To prove (2.45) it suffices to show that for every subsequence $\{n_1\} \subset \{n\}$ there exists a further subsequence $\{n_2\} \subset \{n_1\}$ such that (2.45) holds along $\{n_2\}$. Choose any $\{n_1\} \subset \{n\}$. Since (2.44) holds along $\{n_1\}$, by Theorem 5 there exist a subsequence $\{n_2\} \subset \{n_1\}$ and two functions ψ_1^* and ψ_2^* with the properties given in Theorem 5 such that (1.20) holds at all continuity points of ψ_1^* and ψ_2^*, respectively, and (1.21) holds for some $\delta \geq 0$. An argument based on a theorem of Sato [32] very much like that given in Theorem 3 of Mori [28] shows that (2.44) forces $\psi_1^* = \psi_2^* \equiv 0$. Thus by Theorem 5, $\delta > 0$, which yields

$$|\psi_j(n_2, \lambda)| \to 0 \qquad \text{for all } 0 < \lambda < \infty,\ j = 1, 2,$$

as $n_2 \to \infty$. This in turn implies that (2.45) holds along $\{n_2\}$. Hence (2.45) is valid along the entire sequence $\{n\}$. Of course (2.45) is nothing but

$$n^{-1/2}H(\lambda/n)/\sigma(1/n) \to 0 \qquad \text{for all } 0 < \lambda < \infty \tag{2.46}$$

as $n \to \infty$. It is easy to show that (2.46) is equivalent to (1.26a).

Now assume that (1.26a) holds. By Theorem 1(i) it is enough to show that as $n \to \infty$,

$$\psi_j(n,\lambda) \to 0 \qquad \text{for all } 0 < \lambda < \infty,\ j = 1,2.$$

But obviously

$$|\psi_j(n,\lambda)| \le n^{-1/2}H(\lambda/n)/\sigma(1/n)$$

which by (1.26a) converges to zero. Thus (2.44) holds with $A_n \equiv a(n)$, $B_n \equiv n\mu_{m,k}(n)$, $\mu = 0$, and $\sigma = 1$. Hence we have (1.25).

Since condition (1.26a) is independent of m and k, the equivalency of (1.27) is trivial. □

Proof of Corollary 2. First note that (1.29a) is equivalent to

$$\liminf_{n\to\infty} n^{-1/2}H(\lambda/n)/\sigma(1/n) = 0 \qquad \text{for all } 0 < \lambda < \infty. \tag{2.47}$$

Applying the same argument as given in the proof of Corollary 1, we know that (1.28) holds if and only if there exists a subsequence $\{n_1\}$ of $\{n\}$ such that

$$\lim_{n_1\to\infty} n_1^{-1/2}H(\lambda/n_1)/\sigma(1/n_1) = 0 \qquad \text{for all } 0 < \lambda < \infty. \tag{2.48}$$

Obviously, (2.48) implies (2.47) and hence (1.29a). To complete the proof we need only show that (2.47) implies the existence of a subsequence $\{n_1\}$ of $\{n\}$ such that (2.48) holds.

Assume (2.47) and set

$$J_1 = \{n\colon n^{-1/2}H(1/(2n))/\sigma(1/n) \le \tfrac{1}{2}\}$$

and for $k \ge 2$,

$$J_k = \{n\colon n^{-1/2}H(1/(n2^k))/\sigma(1/n) \le 1/2^k \text{ and } n \in J_{k-1}\}.$$

We observe that by (2.47), J_k is necessarily an infinite set of positive integers for each $k \ge 1$. Let m_1 be the smallest integer in J_1, m_2 be the smallest integer in J_2 greater than m_1, and so on, such that m_{k+1} is the smallest integer in J_{k+1} greater than m_k, etc. Since $m_j \in J_k$ for $j \ge k$, we have

$$m_j^{-1/2}H\big(1/(2^k m_j)\big)/\sigma(1/m_j) \le 2^{-k} \qquad \text{for all } j \ge k.$$

Choose any $\lambda \in (0, \infty)$ and let $k \ge 1$ be such that $2^{-k} < \lambda$. We have

$$\limsup_{j\to\infty} \frac{H(\lambda/m_j)}{m_j^{1/2}\sigma(1/m_j)} \le \limsup_{j\to\infty} \frac{H\big(1/(2^k m_j)\big)}{m_j^{1/2}\sigma(1/m_j)} \le 2^{-k}.$$

Since k can be chosen arbitrarily large,

$$\lim_{j\to\infty} m_j^{-1/2}H(\lambda/m_j)/\sigma(1/m_j) = 0.$$

This completes the proof of the corollary since the equivalency of (1.30) is trivial again. □

Proof of Corollary 3. First assume (1.31). This means, there exist $A_n > 0$ and B_n such that

$$A_n^{-1}\left\{\sum_{j=1}^{n} X_j - B_n\right\} \to_D W \qquad \text{as } n \to \infty,$$

where W is a non-degenerate, non-normal random variable. Then by using Theorem 5, the assumption that $F \notin D(2)$, and the fact following from Theorem 3(ii) that the ψ^* functions of (1.20) obtained along a subsequence $\{n_2\}$ of an arbitrary subsequence $\{n_1\} \subset \{n\}$ are uniquely determined and hence are the same along all these subsequences, we obtain that there must exist ψ_1^* and ψ_2^* as described in Theorem 5 such that

$$-Q(\lambda/n+)/A_n \to -\psi_1^*(\lambda) \qquad \text{and} \qquad Q(1-\lambda/n)/A_n \to -\psi_2^*(\lambda) \tag{2.49}$$

as $n \to \infty$ at every continuity point λ of ψ_1^* and ψ_2^*, respectively, and where at least one of ψ_1^* and ψ_2^* is not identically zero.

As a first case, assume that neither ψ_1^* nor ψ_2^* is identically zero. This implies that there exists a continuity point $\lambda_0 > 0$ of both ψ_1^* and ψ_2^* such that $\psi_j^*(\lambda_0) < 0$, $j = 1, 2$, and

$$Q(\lambda/n+)/Q(\lambda_0/n+) \to \psi_1^*(\lambda)/\psi_1^*(\lambda_0),$$
$$Q(1-\lambda/n)/Q(1-\lambda_0/n) \to \psi_2^*(\lambda)/\psi_2^*(\lambda_0)$$

at every continuity point $\lambda \in (0, \infty)$ of ψ_1^* and ψ_2^*, respectively, as $n \to \infty$. Using monotonicity of Q and the continuity of the ψ^* functions at λ and λ_0, a standard argument based on the integer part function $[\cdot]$ gives that as

$u \downarrow 0$,

$$\begin{aligned} Q(\lambda u+)/Q(\lambda_0 u+) &\to \psi_1^*(\lambda)/\psi_1^*(\lambda_0), \\ Q(1-\lambda u)/Q(1-\lambda_0 u) &\to \psi_2^*(\lambda)/\psi_2^*(\lambda_0). \end{aligned} \tag{2.50}$$

By the characterization theorem of regularly varying functions (cf. [1 or 34]) this implies that there exist constants $-\infty < \nu_1, \nu_2 < \infty$ such that

$$\begin{aligned} &\psi_1^*(\lambda) = \psi_1^*(\lambda_0)(\lambda/\lambda_0)^{-\nu_1}, \qquad \psi_2^*(\lambda) = \psi_2^*(\lambda_0)(\lambda/\lambda_0)^{-\nu_2}, \\ &-Q(s+) = s^{-\nu_1}L_1(s), \qquad Q(1-s) = s^{-\nu_2}L_2(s), \end{aligned} \tag{2.51}$$

where the functions L_1 and L_2 are non-negative on $(0,1)$ and are slowly varying at zero. Note that Theorem 5 also implies (1.19) and this square-integrability forces $\frac{1}{2} < \nu_1, \nu_2 < \infty$. Moreover, the relations in (2.49) and (2.51) imply the asymptotic equality

$$A_n \sim -\left(\psi_j^*(\lambda_0)\lambda_0^{\nu_j}\right)^{-1} L_j(1/n) n^{\nu_j}$$

for both $j = 1$ and $j = 2$. This can only happen if $\nu_1 = \nu_2 = \nu$. Now, taking the ratio of the two limiting relations in (2.49), using the already developed relations in (2.51) with $\nu_1 = \nu_2 = \nu$ and using the argument that has led to (2.50) we arrive at $-Q(s+) \sim cQ(1-s)$ as $s \downarrow 0$, where $c = \psi_1^*(\lambda_0)/\psi_2^*(\lambda_0) > 0$. By (2.51) this means that $L_1(s) \sim cL_2(s)$ as $s \downarrow 0$. Then, introducing for example $L(s) = (L_1(s) + L_2(s))/2$, $\delta_1 = 2/(1 + c^{-1})$, and $\delta_2 = 2/(1 + c)$, we have

$$-Q(s+) = s^{-\nu}L(s)(\delta_1 + o(1)), \qquad Q(1-s) = s^{-\nu}L(s)(\delta_2 + o(1))$$

as $s \downarrow 0$. Thus we can set $\nu = 1/\alpha$ with $0 < \alpha < 2$ and we see that (1.32) holds in the present first case. The second case when $\psi_1^* \equiv 0$ and $\psi_2^* \not\equiv 0$ or $\psi_2^* \equiv 0$ and $\psi_1^* \not\equiv 0$ can be handled similarly.

Next assume that (1.32) holds. Choose $A_n \equiv n^{1/\alpha}L(1/n)$ and $B_n \equiv n\mu_{0,0}(n)$. Then, since L is slowly varying at zero,

$$Q\left(\frac{s}{n}+\right)\Big/A_n \to -\delta_1 s^{-1/\alpha} \quad \text{and} \quad -Q\left(1-\frac{s}{n}\right)\Big/A_n \to -\delta_2 s^{-1/\alpha} \tag{2.52}$$

as $n \to \infty$ for every $s \in (0, \infty)$. Also, Lemma 1 of S. Csörgő, Horváth, and Mason [5] implies that, as $n \to \infty$,

$$n\sigma^2(1/n, 1-1/n) \sim \left(2\left(\delta_1^2 + \delta_2^2\right)/(2-\alpha)\right)L^2(1/n)n^{2/\alpha} \tag{2.53}$$

and that for any sequence of integers r_n such that $r_n \to \infty$, $r_n/n \to 0$,

$$n\sigma^2(r_n/n, 1-r_n/n) \sim \left(2\left(\delta_1^2 + \delta_2^2\right)/(2-\alpha)\right)L^2(r_n/n)n^{2/\alpha}r_n^{1-2/\alpha},$$

which and Lemma 2 in [5] in turn imply that

$$\sigma^2(r_n/n)/\sigma^2(1/n) \to 0 \qquad \text{as } n \to \infty. \tag{2.54}$$

Since by (2.53)

$$a(n)/A_n \to \delta = \left(2(\delta_1^2 + \delta_2^2)/(2-\alpha)\right)^{1/2} > 0,$$

(2.52), (2.54), and Theorem 1 imply that we have (1.34) and hence, in the special case $m = k = 0$, (1.33).

Since the implication (1.33) $\Rightarrow$ (1.31) is trivial, the corollary is proved. □

Proof of Corollary 4. Part (i) follows directly from Corollary 1 and the convergence of types theorem. Concerning part (ii), if (1.32) holds with $L \equiv 1$ and the given α then $F \in DN(\alpha)$ by Corollary 3. The reverse implication follows by an appropriate simplification of the first part of the proof of Corollary 3. □

Proof of Corollary 5. Suppose $F \in D_p(\alpha)$ for an $\alpha \in (0,2)$. By Theorem 3 (see also the remark following Corollary 3) this means that there exists a subsequence $\{n_1\} \subset \{n\}$ and constants $\bar{c}_1, \bar{c}_2 \geq 0$, $\bar{c}_1 + \bar{c}_2 > 0$, and d such that for some constants $A_{n_1} > 0$ and B_{n_1} and for $\bar{\psi}_j^{(\alpha)}(s) = -\bar{c}_j s^{-1/\alpha}$, $0 < s < \infty$, $j = 1,2$, we have

$$A_{n_1}^{-1}\{S_{n_1}(0,0) - B_{n_1}\} \to_D V_{0,0}\left(\bar{\psi}_1^{(\alpha)}, \bar{\psi}_2^{(\alpha)}, 0\right) + d \qquad \text{as } n_1 \to \infty.$$

Theorems 5 and 3 (through the remark again, in view of the uniqueness of the $\psi_j^{(\alpha)}$ functions in the representation of a stable law) then together imply the existence of $\{n_2\} \subset \{n_1\}$ and a constant $\delta > 0$ such that $\psi_j(n_2, s) \to \psi_j^{(\alpha)}(s)$, $0 < s < \infty$, $j = 1,2$, where the involved constants are $c_j = \delta \bar{c}_j$, $j = 1,2$, and

$$\left(a(n_2)\right)^{-1}\{S_{n_2}(0,0) - B_{n_2}\} \to_D V_{0,0}\left(\psi_1^{(\alpha)}, \psi_2^{(\alpha)}, 0\right) + \delta d$$

as $n_2 \to \infty$. But from Theorem 1 there exists an $\{n_3\} \subset \{n_2\}$ such that

$$\left(a(n_3)\right)^{-1}\{S_{n_3}(0,0) - n_3\mu_{0,0}(n_3)\} \to_D V_{0,0}\left(\psi_1^{(\alpha)}, \psi_2^{(\alpha)}, \sigma\right)$$

as $n_3 \to \infty$, where

$$\sigma = \lim_{n_3 \to \infty} \sigma\left(r_{n_3}/n_3\right)/\sigma(1/n_3),$$

and the convergence of types theorem [14, p. 40] forces $\sigma = 0$. This proves the necessity. The sufficiency of the condition and the statement concerning the lightly trimmed sums follow directly from Theorem 1.

It remains to show that $D_p(\alpha)$ is wider than $D(\alpha)$ for each $\alpha \in (0,2)$. This we demonstrate by a direct construction.

The following construction of a function Φ is borrowed from Seneta [33], where it is attributed to an unknown referee. Let $\{\theta_n\}$ be a sequence of positive constants such that

$$\theta_n \uparrow \infty \qquad \text{and} \qquad \theta_{n+1}/\theta_n \to \infty \qquad \text{as } n \to \infty. \tag{2.55}$$

Let n_0 be such that $\theta_{n+1}/\theta_n > 4$ for $n \geq n_0$ and set $\gamma_n = (\theta_n \theta_{n+1})^{1/2}$. Define the continuous function Φ on $[2\gamma_{n_0}, \infty)$ by setting

$$\Phi(x) = \begin{cases} 1, & x \in K := \left[2\gamma_{n_0}, \infty\right) \setminus \bigcup\limits_{r=n_0+1}^{\infty} \left(2^{-1}\gamma_r, 2\gamma_r\right), \\ \dfrac{2}{\gamma_n} x, & x \in \left[2^{-1}\gamma_n, \gamma_n\right], \qquad n \geq n_0 + 1, \\ -\dfrac{1}{\gamma_n} x + 3, & x \in \left(\gamma_n, 2\gamma_n\right], \qquad n \geq n_0 + 1. \end{cases}$$

Then for any fixed $t \in (0, \infty)$ we have

$$\Phi(\theta_n t) = 1 \qquad \text{for all large enough } n, \tag{2.56}$$

but Φ is not slowly varying at infinity because $\Phi(2\gamma_n)/\Phi(\gamma_n) = \frac{1}{2}$ and $\Phi(\gamma_n)/\Phi(\gamma_n/2) = 2$ for all $n \geq n_0 + 1$. Define the continuous function F by

$$F(x) = 1 - x^{-\alpha}\Phi(x), \qquad x \geq 2\gamma_{n_0}.$$

Obviously $F(x) \to 1$ as $x \to \infty$, F is differentiable except in the countably many isolated points $\gamma_n/2$, γ_n, $2\gamma_n$, $n \geq n_0 + 1$, and an elementary computation shows that for $\alpha \in (1,2)$ we have $F'(x) > 0$ whenever $F'(x)$ exists.

From now on we assume that $1 < \alpha < 2$. Thus F is strictly increasing on $[2\gamma_{n_0}, \infty)$ and therefore $1 - F$ can be considered as the upper tail of a distribution function defined on the whole real line. Indeed, from now on we let F denote a distribution function symmetric about zero, whose restriction to $[2\gamma_{n_0}, \infty)$ agrees with the above F. Since Φ is not slowly varying, inversion and Corollary 3 show that $F \notin D(\alpha)$.

In the following we show that the sequence $\{\theta_n\}$ can be chosen so that the corresponding F belongs to $D_p(\alpha)$. To this end, let $\{l_n\}$ be a sequence of positive integers satisfying (2.55) and set $\theta_n = l_n^{1/\alpha}$. Then $\{\theta_n\}$ also satisfies (2.55). For the symmetric F belonging to $\{\theta_n\}$ we have $F(x) = 1 - x^{-\alpha}$ for all large x if and only if $\Phi(x) = 1$, so that for the quantile function Q pertaining to F we get that $Q(1-u) = u^{-1/\alpha}$ for all small

$u > 0$ if and only if $\Phi(u^{-1/\alpha}) = 1$. For every $0 < s < \infty$ we have by (2.56) that $\Phi((s/l_n)^{-1/\alpha}) = \Phi(s^{-1/\alpha}\theta_n) = 1$ for all large n, so that for all large enough n,

$$Q(1 - s/l_n) = l_n^{1/\alpha} s^{-1/\alpha}. \tag{2.57}$$

Using the well-known representation, for $0 < s < \frac{1}{2}$,

$$\begin{aligned}\sigma^2(s) &= s\{Q^2(s) + Q^2(1-s)\} + S^2(s) \\ &\quad -\left(s\{Q(s) + Q(1-s)\} + \int_s^{1-s} Q(u)\,du\right)^2 \end{aligned}\tag{2.58}$$

and the symmetry of Q about $\frac{1}{2}$, we obtain for each $0 < s < \frac{1}{2}$ that

$$\sigma^2(s) = 2sQ^2(1-s) + 2\int_s^{1/2} Q^2(1-u)\,du.$$

Taking into account that

$$Q(1-s) \to \infty \qquad \text{and} \qquad \int_s^{1/2} Q^2(1-u)\,du \to \infty \qquad \text{as } s \downarrow 0,$$

this leads to

$$\sigma^2(s) \sim 2sQ^2(1-s) + 2\int_s^{s_0} Q^2(1-u)\,du \qquad \text{as } s \downarrow 0 \tag{2.59}$$

for any fixed $0 < s_0 < \frac{1}{2}$. Since for all large x we have

$$1 - F_1(x) := x^{-\alpha} \le x^{-\alpha}\Phi(x) = 1 - F(x) \le 2x^{-\alpha} =: 1 - F_2(x),$$

denoting the quantile and σ^2-functions pertaining to the distribution functions F_1 and F_2 symmetric about zero by Q_1, Q_2 and σ_1^2, σ_2^2, respectively, we obtain that for all small $u > 0$,

$$u^{-1/\alpha} = Q_1(1-u) \le Q(1-u) \le Q_2(1-u) = 2^{1/\alpha} u^{-1/\alpha}.$$

By (2.59) this yields

$$(1 + o(1))\sigma_1^2(s) \le \sigma^2(s) \le (1 + o(1))\sigma_2^2(s) \qquad \text{as } s \downarrow 0, \tag{2.60}$$

where

$$\sigma_1^2(s) \sim \frac{4}{2-\alpha} s^{1-2/\alpha} \qquad \text{and} \qquad \sigma_2^2(s) \sim 2^{2/\alpha} \frac{4}{2-\alpha} s^{1-2/\alpha} \qquad \text{as } s \downarrow 0. \tag{2.61}$$

This gives, as $n \to \infty$,

$$(1 + o(1))\left(\frac{4}{2-\alpha}\right)^{1/2} l_n^{1/\alpha} \le l_n^{1/2}\sigma(1/l_n) \le (1 + o(1))\left(2^{2/\alpha}\frac{4}{2-\alpha}\right)^{1/2} l_n^{1/\alpha},$$

which implies the existence of a subsequence $\{l_{n'}\} \subset \{l_n\}$ and a constant c with $2/(2-\alpha)^{1/2} \le c^{-1} \le 2^{1/\alpha}2/(2-\alpha)^{1/2}$ such that

$$l_{n'}^{1/2}\sigma(1/l_{n'})/l_{n'}^{1/\alpha} \to c^{-1} \qquad \text{as } n' \to \infty.$$

By (2.57) and symmetry we therefore obtain that as $n' \to \infty$,

$$\psi_j(l_{n'}, s) \to -cs^{-1/\alpha}, \qquad 0 < s < \infty,\ j = 1,2.$$

This means that the first part of the condition in Corollary 5 is satisfied along the presently chosen $\{n_1\} = \{l_{n'}\}$. On the other hand, for *any* sequence $\{r_{l_{n'}}\}$ of positive integers such that

$$r_{l_{n'}} \to \infty \qquad \text{and} \qquad r_{l_{n'}}/l_{n'} \to 0 \qquad \text{as } n' \to \infty$$

we have by (2.60) and (2.61) that

$$\begin{aligned}\frac{\sigma^2(r_{l_{n'}}/l_{n'})}{\sigma^2(1/l_{n'})} &\le (1 + o(1))2^{2/\alpha}\frac{(r_{l_{n'}}/l_{n'})^{1-2/\alpha}}{(1/l_{n'})^{1-2/\alpha}} \\ &= (1 + o(1))2^{2/\alpha}r_{l_{n'}}^{1-2/\alpha} \to 0\end{aligned}$$

as $n' \to \infty$, so that the second part of the condition in Corollary 5 is also satisfied. This proves $F \in D_p(\alpha)$.

To deal with the case $0 < \alpha \le 1$, one has to modify the function Φ. One possible choice is

$$\Phi(x) = \begin{cases} 1, & x \in K, \\ \dfrac{\alpha}{\gamma_n}x + 1 - \dfrac{\alpha}{2}, & x \in \left[\dfrac{\gamma_n}{2}, \gamma_n\right], \qquad n \ge n_0 + 1, \\ -\dfrac{\alpha}{2\gamma_n}x + 1 + \alpha, & x \in (\gamma_n, 2\gamma_n], \qquad n \ge n_0 + 1. \end{cases}$$

Then (2.56) remains true, Φ is not slowly varying because $\Phi(2\gamma_n)/\Phi(\gamma_n) = 1/(1 + \alpha/2)$ and $\Phi(\gamma_n)/\Phi(\gamma_n/2) = 1 + \alpha/2$, $n \ge n_0 + 1$, $F'(x) > 0$ for each $x \ge 2\gamma_{n_0}$ not equal to $\gamma_n/2$, γ_n, $2\gamma_n$, $n \ge n_0 + 1$. It is exactly this last property that requires the modification. Except for the change of all the constants, everything that we did above remains valid from here on. Thus the corollary is completely proven. □

Proof of Corollary 6. The "if" part is a direct consequence of Theorem 5, Theorem 3(ii), and Theorem 4 applied in this order; i.e., due to the uniqueness of the Lévy representation of the infinitely divisible law, (1.20) and (1.21) must hold along the sequence $\{n_1\}$ and if $\delta > 0$ in (1.21) then the sequence $\{r_{n_1}\}$ in Theorem 1 must already satisfy (1.9). Subsequently the random variable $W_{m,k}$ must be of the stated form for each $m \geq 0$ and $k \geq 0$.

The "only if" part follows by a straightforward modification of the proof of Theorem 4 of Mori [28] using the representation of the characteristic function of $V_{m,k}$ given in Theorem 4. □

Proof of Corollary 7. First assume that $F \in D_p^{(m,k)}$ for some $m \geq 0$ and $k \geq 0$. By Theorem 5 there exist a subsequence $\{n_1\} \subset \{n\}$, $A_{n_1} > 0$, and a non-increasing right-continuous function ϕ on $(0, \infty)$, strictly positive near zero such that

$$n_1^{-1/2} H(\lambda/n_1)/A_{n_1} \to \phi(\lambda) \qquad \text{as } n_1 \to \infty \tag{2.62}$$

at every continuity point λ of ϕ. Choose any continuity point $\lambda_0 < 1$ of ϕ such that $\phi(\lambda_0) > 0$. We see that (2.62) implies that

$$H(\lambda/n_1)/H(\lambda_0/n_1) \to \phi(\lambda)/\phi(\lambda_0) \qquad \text{as } n_1 \to \infty$$

at all continuity points λ of ϕ. This implies (1.38).

Now assume (1.38). By the Helly–Bray theorem choose a subsequence $\{n_2\} \subset \{n_1\}$ and a function ϕ with the above properties such that

$$H(\lambda/n_2)/H(\lambda_0/n_2) \to \phi(\lambda) \tag{2.63}$$

at every continuity point λ of ϕ. Note that necessarily $\phi(\lambda) \geq 1$ for all $0 < \lambda \leq \lambda_0$. Since $F \notin D_p(2)$, there must exist a continuity point λ^*, $0 < \lambda^* < \lambda_0$, and a subsequence $\{n_3\} \subset \{n_2\}$ such that

$$\lim_{n_3 \to \infty} n_3^{-1/2} H(\lambda^*/n_3)/\sigma(1/n_3) = d, \tag{2.64}$$

where $0 < d \leq \infty$. This follows from Corollary 2. From (2.63) we obtain

$$H(\lambda/n_3)/H(\lambda^*/n_3) \to \phi^*(\lambda) := \phi(\lambda)/\phi(\lambda^*) \qquad \text{as } n_3 \to \infty \tag{2.65}$$

at every continuity point of ϕ^*. Let $A_{n_3} = n_3^{-1/2} H(\lambda^*/n_3)$. Now from (2.64) we get

$$\sigma(1/n_3)/A_{n_3} \to c := d^{-1}, \tag{2.66}$$

where $0 \leq c < \infty$. Since $\phi^*(\lambda) \geq 1$ for $0 < \lambda \leq \lambda^*$, $\phi^* \not\equiv 0$. Hence (2.65), (2.66), and Theorem 4 imply (1.37). That (1.39) is also equivalent to the other two statements is trivial. □

Proof of Corollary 8. Corollaries 2 and 7 together imply the equivalency of (1.37) and (1.39). Also, it follows easily from these two corollaries that (1.40a) is sufficient for (1.37). Therefore, it suffices to show that if $F \in D_p$ then (1.40a) holds.

It follows from the proof of Corollary 2 (see (2.48)) that we can assume that $F \notin D_p(2)$. Also, without loss of generality we can assume that $H(s)$ eventually increases as $s \downarrow 0$. From Corollary 7 we have (1.38). For $n_1 \in \{n_1\}$ set $n_2 = [n_1/\lambda_0] + 1$. We have for all $\lambda \in (0, \infty)$,

$$\begin{aligned}&\limsup_{n_2 \to \infty} n_2^{-1/2} H(\lambda/n_2)/\sigma(1/n_2)\\ &\quad \le \limsup_{n_2 \to \infty} n_2^{-1/2} H(1/n_2)/\sigma(1/n_2) \limsup_{n_2 \to \infty} H(\lambda/n_2)/H(1/n_2).\end{aligned}$$

By Lemma 2.4 the first lim sup here is finite. Since for all n_1 sufficiently large

$$\lambda_0/(2n_2) \le 1/n_2 = \left([n_1/\lambda_0] + 1\right)^{-1} \le \lambda_0/n_1,$$

we see by (1.38) that the second lim sup is less than or equal to

$$\limsup_{n_2 \to \infty} H(\lambda_1/n_2)/H(\lambda_0/n_2) < \infty,$$

where $\lambda_1 = \lambda\lambda_0/2$. □

Proof of Corollary 10. Fix $m \ge 0$ and $k \ge 0$ arbitrarily. We know from Theorems 4 and 5 that $F \in SC(m, k)$ if and only if there exists a sequence of positive constants $\tau(n)$ such that for every subsequence $\{n_1\} \subset \{n\}$ there exist a further subsequence $\{m\} \subset \{n_1\}$, a right-continuous, non-negative, non-increasing function ϕ defined on $(0, \infty)$, and a constant $\delta \in [0, \infty)$ such that

$$\phi(m, \lambda) := \frac{H(\lambda/m)}{m^{1/2}\tau(m)} \to \phi(\lambda) \qquad \text{as } m \to \infty \tag{2.67}$$

at every continuity point λ of ϕ and

$$\sigma(1/m)/\tau(m) \to \delta \qquad \text{as } m \to \infty, \tag{2.68}$$

where $\phi \not\equiv 0$ and $\phi(\lambda) = 0$ for all $\lambda \ge 1$ if $\delta = 0$.

Since condition (1.42a) is clearly equivalent to

$$\limsup_{n \to \infty} \frac{H(\lambda/n)}{n^{1/2}\sigma(1/n)} < \infty \qquad \text{for all } 0 < \lambda < 1, \tag{2.69}$$

that (1.42a) implies (1.41) follows at once from Theorem 1.

Now we turn to the necessity implication (1.41) $\Rightarrow$ (2.69). The key step towards this aim is the following result.

LEMMA 2.14. *If $F \in SC(m, k)$ for some $m \geq 0$ and $k \geq 0$, then the case $\delta = 0$ can never happen in* (2.68).

Proof. Choose a subsequence $\{m\} \subset \{n\}$ such that (2.67) and (2.68) hold and assume that $\delta = 0$. Then necessarily

$$\phi(\lambda) = 0 \qquad \text{for all } \lambda \geq 1, \tag{2.70}$$

but $\phi \not\equiv 0$. Select a sequence of continuity points $c_1, c_2, \dots$ of ϕ such that $1 > c_1 > c_2 > \cdots$, $c_j \to 0$ as $j \to \infty$ and $\phi(c_j) > 0$ for all $j \geq 1$. For all $j \geq 1$ consider the sequences $m_j = [mc_j]$, $m \geq c_j^{-1}$. We claim that there exist a further subsequence $\{m'\}$ of $\{m\}$ and constants d_j such that for all $j \geq 1$,

$$\frac{(m_j')^{1/2}\tau(m_j')}{(m')^{1/2}\tau(m')} \to d_j \qquad \text{as } m' \to \infty \text{ and } 0 < d_j < \infty. \tag{2.71}$$

To prove this claim, by a diagonal procedure choose any subsequence $\{m'\} \subset \{m\}$ such that the limits

$$\lim_{m' \to \infty} \left\{(m_j')^{1/2}\tau(m_j')\right\} \Big/ \left\{(m')^{1/2}\tau(m')\right\} = d_j$$

exist for all $j \geq 1$. Then necessarily $0 \leq d_j \leq \infty$ for all $j \geq 1$, and to complete the proof of the claim it is enough to show that for any $j \geq 1$, the cases $d_j = 0$ and $d_j = \infty$ cannot happen.

Suppose first that $d_j = 0$ for some $j \geq 1$. Then for this j and for all $0 < s < c_j^2$ we have

$$\liminf_{m' \to \infty} \frac{H(s/m_j')}{(m_j')^{1/2}\tau(m_j')} = \liminf_{m' \to \infty} \frac{H\left(sm'/(m_j'm')\right)}{(m')^{1/2}\tau(m')} \frac{(m')^{1/2}\tau(m')}{(m_j')^{1/2}\tau(m_j')}.$$

Since by (2.67),

$$\lim_{m' \to \infty} H\left(sm'/(m_j'm')\right) \Big/ \left\{(m')^{1/2}\tau(m')\right\} = \phi\left(sc_j^{-1}\right) > 0,$$

$d_j = 0$ implies that

$$\lim_{m' \to \infty} H(s/m_j')/\left\{(m_j')^{1/2}\tau(m_j')\right\} = \infty \qquad \text{for all } 0 < s < c_j^2,$$

contradicting the stochastic compactness assumption that $F \in SC(m, k)$.

Next suppose that $d_j = \infty$ for some $j \geq 1$. Let $s \in (0, \infty)$ be arbitrary and choose a continuity point s' of ϕ in (2.67) such that $s' < s/c_j$. Then, writing lim sup instead of lim inf in the above equality we obtain

$$\limsup_{m' \to \infty} \frac{H(s/m'_j)}{(m'_j)^{1/2}\tau(m'_j)} \leq \limsup_{m' \to \infty} \frac{H(s'/m')}{(m')^{1/2}\tau(m')} \frac{(m')^{1/2}\tau(m')}{(m'_j)^{1/2}\tau(m'_j)}.$$

Hence

$$\lim_{m' \to \infty} H(s/m'_j) \Big/ \left\{(m'_j)^{1/2}\tau(m'_j)\right\} = 0 \qquad \text{for all } 0 < s < \infty. \quad (2.72)$$

However, since $\sigma(1/m'_j) \leq \sigma(1/m')$, (2.68) implies that $\tau(m')/\sigma(1/m') \to \infty$ and $\tau(m'_j)/\tau(m') \to \infty$ as $m' \to \infty$. Thus $\sigma(1/m'_j)/\tau(m'_j) \to 0$ as $m' \to \infty$. By Theorem 5 this last relation together with (2.72) also contradicts the stochastic compactness condition that $F \in SC(m, k)$.

Thus we have completely proved the claim in (2.71).

Now let $\{m'\}$ be a subsequence of $\{m\}$ (for which (2.67) and (2.68) hold with $\delta = 0$) such that (2.71) holds for all $j \geq 1$. Notice that for all $j \geq 1$,

$$\phi(m'_j, \lambda) \to \phi_j(\lambda) := d_j^{-1}\phi(\lambda/c_j) \qquad \text{as } m' \to \infty$$

at every point $\lambda > 0$ such that λ/c_j is a continuity point of ϕ. Note that by (2.70), $\phi_j(\lambda) = 0$ for all $c_j \leq \lambda < \infty$, $j \geq 1$. Also, since $\tau(m'_j)/\tau(m') \to d_j/c_j$ for all $j \geq 1$ as $m' \to \infty$, where all these limits are strictly positive and finite, the relation (2.68) as applied along $\{m'\}$ and with $\delta = 0$ forces $\sigma(1/m'_j)/\tau(m'_j) \to 0$ as $m' \to \infty$ for all $j \geq 1$.

Therefore, by another diagonal selection procedure we can find a subsequence $\{m''\}$ of $\{m'\}$ such that $\phi_{m''}(\lambda) \to 0$ for all $0 < \lambda < \infty$ and $\sigma(1/m'')/\tau(m'') \to 0$ as $m'' \to \infty$. This contradicts the assumption that $F \in SC(m, k)$, and thus the lemma is completely proved. □

Returning to the proof of the implication (1.41) ⇒ (2.69), note that Lemma 2.14 easily implies that if (1.41) holds then for the sequence $\{\tau(n)\}$ figuring in (2.67) and (2.68) we have

$$0 < \liminf_{n \to \infty} \frac{\sigma(1/n)}{\tau(n)} \leq \limsup_{n \to \infty} \frac{\sigma(1/n)}{\tau(n)} < \infty. \quad (2.73)$$

Assume now (1.41) and that, contrary to (2.69), there exists a $\lambda \in (0, 1)$ such that

$$\limsup_{n \to \infty} H(\lambda/n) \Big/ \left\{ n^{1/2}\sigma(1/n) \right\} = \infty.$$

Then we can find a subsequence $\{n_1\} \subset \{n\}$ such that

$$\lim_{n_1 \to \infty} H(\lambda/n_1) \big/ \{ n_1^{1/2} \sigma(1/n_1) \} = \infty,$$

but by (2.73) this implies that $\phi(n_1, \lambda) \to \infty$ as $n_1 \to \infty$, contradicting (1.41).

Again, (1.42) and (1.43) are trivially equivalent.

Finally we show that (1.44) is satisfied for all subsequential limiting random variables $cV_{m,k}(\psi_1, \psi_2, \sigma) + d$, where $c > 0$ and d are constants, if $F \in SC$. As noted before the proof of Corollary 1, this we do by using the fact that conditions (1.42a) and (1.42c) are equivalent. This is proved in the next section, and a small element of the (independent) proof given there is also used here.

By Lemma 2.14 and Theorems 5 and 1 we can assume without loss of generality that $\psi_j(n_i, s) \to \psi_j(s)$ at every continuity point s of ψ_j, $j = 1, 2$, and

$$\sigma^2\big((r_{n_i} + 1)/n_i\big)/\sigma^2(1/n_i) \to \sigma^2$$

as $i \to \infty$, where $\{n_1, n_2, \dots,\}$ is some subsequence of $\{n\}$, in which case $c = 1$ and $d = 0$.

It follows from (1.42c) that for all large enough i,

$$\begin{aligned}
s\{\psi_1^2(n_i, s) &+ \psi_2^2(n_i, s)\} \\
&\le C_1\big(\sigma^2(1/n_i)\big)^{-1} \int_{s/n_i}^{1-s/n_i} Q^2(u)\, du \\
&= C_1\big(\sigma^2(1/n_i)\big)^{-1}\bigg\{ \int_{s/n_i}^{\lambda/n_i} \big(Q^2(u) + Q^2(1-u)\big)\, du \\
&\qquad\qquad + \int_{\lambda/n_i}^{1/2} \big(Q^2(u) + Q^2(1-u)\big)\, du \bigg\} \\
&= C_1\bigg\{ \int_s^{\lambda} \big(\psi_1^2(n_i, t) + \psi_2^2(n_i, t)\big)\, dt + \frac{S^2(\lambda/n_i)\sigma^2(\lambda/n_i)}{\sigma^2(\lambda/n_i)\sigma^2(1/n_i)} \bigg\},
\end{aligned}$$

where C_1 is a finite constant depending only on F and λ is any fixed number greater than s. Assume now that s is a continuity point of both ψ_1 and ψ_2. Since for all $t \in [s, \lambda]$ the integrand in the first integral is majorized by the constant

$$2\{\psi_1^2(s) + \psi_2^2(s)\},$$

and

$$\sigma^2(\lambda/n_i) \le \sigma^2\big((r_{n_i} + 1)/n_i\big)$$

if i is large, letting first $i \to \infty$ and then $\lambda \to \infty$ we obtain

$$s\{\psi_1^2(s) + \psi_2^2(s)\} \le C_1\left\{\int_s^\infty \left(\psi_1^2(t) + \psi_2^2(t)\right) dt + C_2\sigma^2\right\},$$

where

$$C_2 = \limsup_{\lambda\to\infty} \limsup_{i\to\infty} \frac{S^2(\lambda/n_i)}{\sigma^2(\lambda/n_i)} \le \limsup_{s\downarrow 0} S^2(s)/\sigma^2(s) < \infty$$

because of Lemma 3.2 in the next section. This proves (1.44) for common continuity points $s > 0$, from which the general statement follows by right-continuity. Corollary 10 is completely proven. □

Proof of Corollary 11. The equivalency of (1.45), (1.46), and (1.47) is a direct consequence of Theorems 4 and 5.

To show that SSC is a wider class than SC we construct a quantile function Q such that the corresponding F is in $SSC \setminus SC$. It is defined by introducing

$$I_k = \left(2^{-(k+1)^2}, 2^{-k^2}\right], \qquad k = 0, 1, 2, \dots,$$

and setting

$$Q(s) = \begin{cases} 0, & \text{if } \frac{1}{2} < s \le 1, \\ -2, & \text{if } s \in I_1, \\ -2^{(k+1)^2/2}\sigma_{k-1}, & \text{if } s \in I_k,\ k = 2, 3, \dots, \end{cases} \tag{2.74}$$

where

$$\sigma_k^2 = \sigma\left(2^{-(k+1)^2}\right), \qquad k = 1, 2, \dots,$$

in particular, $\sigma_1 = \sigma(\frac{1}{16}) = 1$.

Now we analyze this Q and list its relevant properties. The first one of these follows by a close look at the definition.

PROPERTY 1. For

$$K = \sup_{0 < s \le 1/16} - s^{1/2}Q(s)/\sigma(s)$$

we have $0 < K < \infty$.

Let $\bar{I}_k = [2^{-(k+1)^2}, 2^{-k^2}]$ be the closure of I_k, $k = 0, 1, 2, \dots$.

PROPERTY 2. For each $k \geq 2$, $s \in I_k$, and $0 < \lambda < \infty$ such that $\lambda s \in I_0 \cup I_1 \cup \cdots \cup I_{k-1} \cup \bar{I}_k$ we have

$$-s^{1/2}Q(\lambda s+)/\sigma(s) \leq K.$$

Indeed, for any such s and λ,

$$\begin{aligned} -s^{1/2}Q(\lambda s+)/\sigma(s) &\leq -s^{1/2}Q(2^{-k^2})/\sigma(s) \\ &= -s^{1/2}Q(2^{-(k+1)^2}+)/\sigma(s) \\ &= -s^{1/2}Q(s)/\sigma(s) \leq K. \end{aligned}$$

Introducing now the left-closed, right-open variant $\tilde{I}_k = [2^{-(k+1)^2}, 2^{-k^2})$ of I_k, $k = 0, 1, 2, \ldots$, and the sequence

$$C(k) = 2^{((k+2)^2-(k+1)^2)/2}\sigma_k, \qquad k = 1, 2, \ldots,$$

the following two properties are obvious.

PROPERTY 3. For each $k \geq 2$, $s \in \bar{I}_k$, and $0 < \lambda < 1$ such that $\lambda s \in \tilde{I}_{k+1}$,

$$-s^{1/2}Q(\lambda s+)/\sigma(s) = s^{1/2}2^{(k+2)^2/2}\sigma_k/\sigma(s) \geq 2^{((k+2)^2-(k+1)^2)/2}$$

PROPERTY 4. For each $k \geq 2$, $s \in \bar{I}_k$, and $0 < \lambda < 1$ such that $\lambda s \in \tilde{I}_{k+1}$,

$$-s^{1/2}Q(\lambda s+)/C(k) = \left(2^{(k+1)^2}s\right)^{1/2}$$

PROPERTY 5. Let $\{k_j, j \geq 1\}$ be a sequence of integers, $k_j \geq 1$ for all $j \geq 1$ and $k_j \to \infty$ as $j \to \infty$ and $\{s_{k_j}, j \geq 1\}$ be a sequence of numbers such that $s_{k_j} \in I_{k_j}$, $j \geq 1$, and $2^{(k_j+1)^2}s_{k_j} \to c$ as $j \to \infty$, where $1 \leq c < \infty$. Then for all positive $\lambda \neq c$ we have

$$-s_{k_j}^{1/2}Q(\lambda s_{k_j}+)/C(k_j) \to (\lambda c)^{1/2}I(\lambda < c^{-1}) \qquad \text{as } j \to \infty,$$

where I is the indicator function as earlier.

To show this first assume $0 < \lambda < c^{-1}$. Then for all j sufficiently large,

$$2^{-(k_j+2)^2} < \lambda s_{k_j} < 2^{-(k_j+1)^2}$$

Thus by Property 4, for all large j,

$$-s_{k_j}^{1/2}Q(\lambda s_{k_j}+)/C(k_j) = \left(2^{(k_j+1)^2}\lambda s_{k_j}\right)^{1/2}$$

which by assumption converges to $(\lambda c)^{1/2}$ if $j \to \infty$.

Next assume that $c^{-1} < \lambda < \infty$. In this case, we have for all large j that

$$\lambda s_{k_j} \in I_0 \cup I_1 \cup \cdots \cup I_{k_j-1} \cup \bar{I}_{k_j},$$

so that by Property 2,

$$s_{k_j}^{1/2} Q\left(\lambda s_{k_j}+\right)/\sigma\left(s_{k_j}\right) \leq K.$$

Notice that

$$\sigma\left(s_{k_j}\right)/C\left(k_j\right) \leq \sigma_{k_j}/C\left(k_j\right) = 2^{-((k_j+2)^2-(k_j+1)^2)/2},$$

which converges to zero if $j \to \infty$. Thus, in this case,

$$-s_{k_j}^{1/2} Q\left(\lambda s_{k_j}+\right)/C\left(k_j\right) \to 0 \qquad \text{as } j \to \infty,$$

which proves Property 5.

PROPERTY 6. Let $\{k_j,\ j \geq 1\}$ be as in Property 5 and $\{s_{k_j},\ j \geq 1\}$ be a sequence such that $s_{k_j} \in I_{k_j}$, $j \geq 1$, and $2^{(k_j+1)^2} s_{k_j} \to \infty$ as $j \to \infty$. Then for all $0 < \lambda < \infty$,

$$\limsup_{j \to \infty} \; - s_{k_j}^{1/2} Q\left(\lambda s_{k_j}+\right)/\sigma\left(s_{k_j}\right) \leq K.$$

Indeed, by Property 1 it is sufficient to show this only for $0 < \lambda < 1$. But for any $0 < \lambda < 1$ and for all j sufficiently large we necessarily have

$$s_{k_j} \in I_0 \cup I_1 \cup \cdots \cup I_{k_j-1} \cup \bar{I}_{k_j},$$

and thus by Property 2, for all such j,

$$-s_{k_j}^{1/2} Q\left(\lambda s_{k_j}+\right)/\sigma\left(s_{k_j}\right) \leq K,$$

finishing the proof of this property.

Now we can complete the proof of the present corollary. Notice first that the random variables $X_1, X_2, \ldots,$ with quantile function Q in (2.74) are non-positive and in this case the stochastic compactness condition (1.42a) is equivalent to

$$\limsup_{s \downarrow 0} \; - s^{1/2} Q(\lambda s+)/\sigma(s) < \infty \qquad \text{for all } 0 < \lambda < 1. \quad (2.75)$$

Presently, by Property 3 for all $\lambda \in (0,1)$ there exists a $k_0 = k_0(\lambda)$ such that if $k \geq k_0$ then

$$-2^{-(k+1)^2/2}Q(\lambda 2^{-(k+1)^2})/\sigma(2^{-(k+1)^2}) = 2^{k+3/2},$$

which goes to infinity as $k \to \infty$. Hence (2.75) fails and thus by Corollary 10, $F \notin SC$.

On the other hand, let $\{n_1, n_2, \ldots\}$ be any subsequence of $\{n\}$. For each n_j determine a $k(n_j)$ such that $1/n_j \in I_{k(n_j)}$. Now there exists a subsequence $\{j_l\} \subset \{j\}$ such that

$$2^{(k(n_{j_l})+1)^2} n_{j_l}^{-1} \to c \qquad \text{as } l \to \infty \tag{2.76}$$

for some $1 \leq c \leq \infty$. If $c < \infty$ then for all $0 < \lambda < \infty$, $\lambda \neq c$,

$$n_{j_l}^{-1/2} Q(\lambda/n_{j_l}+)/C(k(n_{j_l})) \to -(\lambda c)^{1/2} I(\lambda < c^{-1})$$

as $l \to \infty$ by Property 5. If $c = \infty$, then by Property 6 there exists a further subsequence $\{m_l\} \subset \{n_{j_l}\}$ and a non-positive, non-decreasing, right-continuous function ψ on $(0, \infty)$ such that

$$m_l^{-1/2}Q(\lambda/m_l+)/\sigma(1/m_l) \to \psi(\lambda) \qquad \text{as } l \to \infty$$

at every continuity point λ of ψ. In either case, by Theorem 2 and Theorem 1, respectively, we can find a subsequence $\{m_j\}$ of the above arbitrary $\{n_j\}$ and sequences of constants $A_{m_j} > 0$ and B_{m_j} such that

$$A_{m_j}^{-1}\{S_{m_j}(0,0) - B_{m_j}\} \to_D W,$$

where W is a non-degenerate infinitely divisible random variable. This means that $F \in SSC$. □

Note that in the above proof the case $c < \infty$ in (2.76) indeed appears for some sequences $\{n_j, j \geq 1\}$. This shows the validity of the statement following Theorem 1 in Section 1 that Theorem 2 is not empty.

Proof of Corollary 12. First assume that $F \in SC$. Then by Theorem 5 there exists a sequence of normalizing constants $A_n > 0$ such that for every subsequence $\{n_1\}$ of $\{n\}$ there exists a further subsequence $\{n_2\} \subset \{n_1\}$ for which (1.19) and (1.20) hold with $\psi_1^* \equiv 0$ and $\psi_2^* = -\phi$ nonidentically zero. An easy argument based on the representation

$$A_n^{-1}X_{n,n} = A_n^{-1}Q(U_{n,n}) = -A_n^{-1}a(n)\psi_2(n, n(1 - U_{n,n})),$$

cf. (2.2), combined with the fact that

$$n(1 - U_{n,n}) \to_p Y_1^{(2)} \qquad \text{as } n \to \infty,$$

shows that as $n_2 \to \infty$,

$$A_{n_2}^{-1} X_{n_2, n_2} \to_p \phi\left(Y_1^{(2)}\right).$$

This implies that $\{X_{n,n}\}$ is stochastically compact.

Conversely, if there exists a sequence of normalizing constants $A_n > 0$ such that $\{A_n^{-1} X_{n,n}\}$ is stochastically compact, then it is routine to verify that for every subsequence $\{n_1\}$ of $\{n\}$ there exists a further subsequence $\{n_2\}$ of $\{n_1\}$ such that (1.20) of Theorem 5 holds with $\psi_1^* \equiv 0$ and $\psi_2^* = -\phi$ not identically zero. This, since $F \notin D_p(2)$, by Corollary 9 forces either Theorem 1 with $\sigma = 0$ or Theorem 2 to be applicable with a non-degenerate limit. Thus $F \in SC$.

The last statement of the corollary follows from Lemma 2.14 and (1.44), or simply from the last statement of Corollary 10. □

3. The Equivalence of the a–b–c Conditions

For the sake of convenience we first note that conditions (1.26a), (1.29a), (1.40a), and (1.42a) are clearly equivalent to

$$\lim_{s \downarrow 0} sG^2(\lambda s)/\sigma^2(s) = 0 \qquad \text{for all } 0 < \lambda < 1; \tag{3.1a}$$

$$\liminf_{s \downarrow 0} sG^2(\lambda s)/\sigma^2(s) = 0 \qquad \text{for all } 0 < \lambda < 1; \tag{3.2a}$$

There exists a subsequence $\{n_1\}$ of $\{n\}$ such that

$$\limsup_{n_1 \to \infty} n_1^{-1} G^2(\lambda/n_1)/\sigma^2(1/n_1) < \infty \qquad \text{for all } 0 < \lambda < 1; \tag{3.3a}$$

$$\limsup_{s \downarrow 0} sG^2(\lambda s)/\sigma^2(s) < \infty \qquad \text{for all } 0 < \lambda < 1; \tag{3.4a}$$

respectively, where $G^2(s) = Q^2(s) + Q^2(1-s)$, $0 < s < 1$, is the function figuring in the c-type conditions.

First we show the equivalences (3.1a) $\Leftrightarrow$ (1.26b), (3.2a) $\Leftrightarrow$ (1.29b), (3.3a) $\Leftrightarrow$ (1.40b), and (3.4a) $\Leftrightarrow$ (1.42b). To this end note that by Lemma 2.4

$$M := \limsup_{s \downarrow 0} sG^2(s)/\sigma^2(s) < \infty. \tag{3.5}$$

In order to prove the four implications b $\Rightarrow$ a, we observe that by (2.29), for any $0 < s < \frac{1}{2}$ and $0 < \lambda < 1$,

$$\begin{aligned} &\sigma^2(\lambda s) - \sigma^2(s) \\ &\quad \geq \frac{\lambda}{2} s\left\{(Q(s) - Q(\lambda s))^2 + (Q(1-\lambda s) - Q(1-s))^2\right\}. \end{aligned} \tag{3.6}$$

If (1.26b) holds, then dividing this by $\sigma^2(s)$ and letting $s \downarrow 0$, we obtain

$$A(\lambda) := \limsup_{s \downarrow 0} \left\{ \left(\frac{s^{1/2}Q(s)}{\sigma(s)} - \frac{s^{1/2}Q(\lambda s)}{\sigma(s)} \right)^2 + \left(\frac{s^{1/2}Q(1-\lambda s)}{\sigma(s)} - \frac{s^{1/2}Q(1-s)}{\sigma(s)} \right)^2 \right\} = 0,$$

which in combination with (3.5) says that for all $0 < \lambda < 1$,

$$sG^2(\lambda s)/\sigma^2(s) = sG^2(s)/\sigma^2(s) + o(s) \le M + o(s) \tag{3.7}$$

as $s \downarrow 0$. Thus by (1.26b) again, for all $0 < \lambda < 1$,

$$\limsup_{t \downarrow 0} tG^2(t)/\sigma^2(t) = \limsup_{s \downarrow 0} \lambda s G^2(\lambda s)/\sigma^2(\lambda s) \le \lambda M,$$

which forces the left side to be zero. This and (3.7) imply (3.1a).

The implication (1.29b) $\Rightarrow$ (3.2a) is proved along the same lines.

If (1.42b) holds, then again dividing the inequality in (3.6) by $\sigma^2(s)$ and letting $s \downarrow 0$, we obtain $A(\lambda) < \infty$ for each $0 < \lambda < 1$, which in combination with (3.5) yields (3.4a).

The implication (1.40b) $\Rightarrow$ (3.3a) is proved in the same way. □

Noting that condition (3.2a) is equivalent to the existence of a sequence $s_m \in (0,1)$ with $s_m \to 0$ as $m \to \infty$ such that for all $0 < \lambda < 1$,

$$\lim_{m \to \infty} s_m \{ Q^2(\lambda s_m) + Q^2(1 - \lambda s_m) \} / \sigma^2(s_m) = 0 \tag{3.8}$$

(cf. the proof of Corollary 2), the reverse four implications (3.1a) $\Rightarrow$ (1.26b), (3.2a) $\Rightarrow$ (1.29b), (3.3a) $\Rightarrow$ (1.40b), and (3.4a) $\Rightarrow$ (1.42b) all follow easily by substituting $t = \lambda s$ into the inequality (2.30). □

Now we turn to the proof of the equivalencies (3.1a) $\Leftrightarrow$ (1.26c), (3.2a) $\Leftrightarrow$ (1.29c), (3.3a) $\Leftrightarrow$ (1.40c), and (3.4a) $\Leftrightarrow$ (1.42c). If $\mathrm{Var}(X) < \infty$, then the limits in (3.1a) and (1.26c) are both zero, and so all these eight implications are automatically true. We may therefore assume throughout that $\mathrm{Var}(X) = \infty$. This implies

$$\lim_{s \downarrow 0} S^2(s) = \infty. \tag{3.9}$$

First we prove the four implications a $\Rightarrow$ c. From the representation in (2.58) we have

$$\sigma^2(s) \le sG^2(s) + S^2(s), \qquad 0 < s < \tfrac{1}{2}. \tag{3.10}$$

As a first case, suppose that there exists a $\delta \in (0, \frac{1}{2})$ such that $Q(u) < 0$ and $Q(1-u) > 0$ for all $0 < u < \delta$. The function $G^2(\cdot)$ is then non-increasing in $(0, \delta)$ and hence

$$S^2(s) \geq \int_s^{2s} G^2(u)\, du \geq sG^2(2s), \qquad 0 < s < \delta/2. \qquad (3.11)$$

Combining this with (3.10), for $0 < s < \delta/2$ we get

$$\sigma^2(2s) \leq 2S^2(s) + S^2(2s) \leq 3S^2(s),$$

from which we obtain for all $0 < \lambda \leq 1$,

$$sG^2(\lambda s)/S^2(s) \leq 3sG^2(\lambda s)/\sigma^2(2s).$$

Using now the a-type condition, we see that the implications (3.1a) $\Rightarrow$ (1.26c), (3.2a) $\Rightarrow$ (1.29c), (3.3a) $\Rightarrow$ (1.40c), and (3.4a) $\Rightarrow$ (1.42c) all follow in the present case.

Consider now the second case when $Q(u) \geq 0$ for all $u \in (0,1)$. Thus from (3.10) and (3.9) we obtain

$$\sigma^2(s) \leq sQ^2(1-s) + S^2(s)(1 + o(1)) \qquad \text{as } s \downarrow 0,$$

and instead of (3.11) we have

$$S^2(s) \geq \int_s^{2s} Q^2(1-u)\, du \geq sQ^2(1-2s), \qquad 0 < s < \tfrac{1}{4}.$$

Thus $\sigma^2(2s) \leq (3 + o(1))S^2(s)$ as $s \downarrow 0$ and the proof can be completed as before. Since the third case, when $Q(1-u) \leq 0$ for all $u \in (0,1)$, is similar to the second, the implications (3.1a) $\Rightarrow$ (1.26c), (3.2a) $\Rightarrow$ (1.29c), (3.3a) $\Rightarrow$ (1.40c), and (3.4a) $\Rightarrow$ (1.42c) are completely proven. □

In the proof of the reverse implications c $\Rightarrow$ a we need the following two lemmas, the first of which follows from the proof of the converse part of Theorem A.2.b in Seneta [34] and which goes back to Karamata (cf. Seneta's remark on page 97 in [34]).

LEMMA 3.1. *Let $c > 0$ be a constant and K: $[c, \infty) \to (0, \infty)$ be a measurable function which is Lebesgue integrable on any finite subinterval of $[c, \infty)$. Suppose that there exists a real constant k such that*

$$\limsup_{x \to \infty} x^{k+1}K(x) \Big/ \int_c^x y^k K(y)\, dy < \infty.$$

Then for all $\lambda \in (0,1)$,

$$\limsup_{x \to \infty} \int_c^{x/\lambda} y^k K(y)\, dy \Big/ \int_c^x y^k K(y)\, dy < \infty.$$

LEMMA 3.2. *For any quantile function Q,*

$$\limsup_{s \downarrow 0} S^2(s)/\sigma^2(s) < \infty.$$

Proof. Again from the representation (2.58), using elementary inequalities, we obtain for any $0 < s < \frac{1}{2}$ that

$$\frac{S^2(s)}{\sigma^2(s)} \leq \left(1 + \left\{sG^2(s)(1-4s) - 2\left(\int_s^{1-s} Q(u)\,du\right)^2\right\} \Big/ S^2(s)\right)^{-1}$$

Since the statement is trivial if $\mathrm{Var}(X) < \infty$, we may assume the opposite. In this case, exactly as in [14, p. 173], one can show that

$$\int_s^{1-s} Q(u)\,du/S(s) \to 0 \qquad \text{as } s \downarrow 0.$$

This implies the lemma. □

Fix a $\lambda \in (0,1)$ and for $s > 0$ small enough write

$$\frac{sG^2(\lambda s)}{\sigma^2(s)} = \frac{1}{\lambda}\frac{\lambda s G^2(\lambda s)}{S^2(\lambda s)}\frac{S^2(s)}{\sigma^2(s)}\frac{S^2(\lambda s)}{S^2(s)}. \tag{3.12}$$

In view of Lemma 3.2 we see that the implications (1.26c) ⇒ (3.1a) and (1.42c) ⇒ (3.4a) will be both proven if we show that

$$(1.42\mathrm{c}) \Rightarrow \limsup_{s \downarrow 0} \frac{S^2(\lambda s)}{S^2(s)} < \infty \qquad \text{for all } 0 < \lambda < 1. \tag{3.13}$$

Set

$$K(x) = G^2(x^{-1}) = Q^2(x^{-1}) + Q^2(1 - x^{-1}), \qquad c \leq x < \infty,$$

where $c \geq 2$ is a constant such that $K(x) > 0$ for $c \leq x < \infty$. For any $0 < s < \frac{1}{2}$ we have

$$S^2(s) = \int_s^{1/c} G^2(u)\,du + \int_{1/c}^{1/2} G^2(u)\,du,$$

which because of (3.9) implies

$$S^2(s) \sim \int_s^{1/c} G^2(u)\,du \qquad \text{as } s \downarrow 0. \tag{3.14}$$

We thus have, by a change of variables and (1.42c), that

$$\limsup_{x \to \infty} x^{-1}K(x) \Big/ \int_c^x y^{-2}K(y)\,dy = \limsup_{s \downarrow 0} sG^2(s)/S^2(s) < \infty.$$

Applying Lemma 3.1 to our special function K with $k = -2$ yields for any $\lambda \in (0,1)$ that

$$\limsup_{x\to\infty} \int_c^{x/\lambda} y^{-2}K(y)\,dy \Big/ \int_c^x y^{-2}K(y)\,dy < \infty.$$

This by a change of variables is equivalent to

$$\limsup_{s\downarrow 0} S^2(\lambda s)/S^2(s) = \limsup_{s\downarrow 0} \int_{\lambda s}^{1/c} G^2(u)\,du \Big/ \int_s^{1/c} G^2(u)\,du < \infty,$$

where the equality follows from (3.14). Thus (3.13) and hence the two implications (1.26c) $\Rightarrow$ (3.1a) and (1.42c) $\Rightarrow$ (3.4a) are completely proven.

Implications (1.29c) $\Rightarrow$ (3.2a) and (1.40c) $\Rightarrow$ (3.3a) follow from writing for any $0 < \lambda < 1$,

$$sG^2(\lambda s)/\sigma^2(s) = \left\{sG^2(\lambda s)/S^2(s)\right\}\left\{S^2(s)/\sigma^2(s)\right\}$$

and then applying Lemma 3.2. □

4. Discussion

Now we discuss the results in Section 1 and their relationship with the existing literature. Theorems 1, 2, 4, and 5 are new. Theorem 3, though likely never stated in the literature, can also be inferred from Ito's well-known representation for a stochastic process with independent increments. Except for the very special case when $F \in D(\alpha)$, $0 < \alpha \leq 2$, treated by S. Csörgő, Horváth, and Mason [5], we are not aware of any other comparable results in the literature explaining the fine asymptotic structure of sums or their lightly trimmed versions with a view of showing which portions of these sums contribute the ingredients of the limiting laws. Certain aspects, however, of the role of the minima and maxima on determining the Lévy measures in the triangular array setup can be found in classical works such as Loève [23a] and [23b].

There are two kinds of light trimming which reveal the influence of extreme terms in the sum on the limiting distribution. One is when a fixed number of terms largest in absolute value are excluded from the whole sum at each stage n. There are many papers on this problem, as a rule, under restrictive initial assumptions; most of these are referenced in some of the papers cited in the present one. Results of this type will be mentioned below only if they have some relevance in connection with the corollary we discuss. This kind of trimming will be referred to as "modulus trimming." The other kind of trimming is when a fixed number of the smallest and the largest order statistics are discarded from the sum at each stage n, with respect to the natural ordering. This is the one that we investigated here. The pioneering paper in this connection is Darling [6] who basically deals with positive random variables with $F \in D(\alpha)$, $\alpha < 2$, when the two kinds of trimming are the same.

$D(2)$ was first characterized independently by Khinchin [17], Lévy [22], and Feller [9] by the well-known criterion on F (cf. [14, p. 172]) that

$$\lim_{x \to \infty} x^2\{F(-x) + 1 - F(x)\} \Big/ \int_{-x}^{x} y^2 \, dF(y) = 0. \tag{4.1}$$

Equivalent conditions all based on F have been given by Feller [10, p. 303] and Maller [25]. A criterion expressed in terms of Q was shown to be equivalent to (4.1) in [3]. It is a combined form of conditions (1.26a) and (1.26c). Condition (1.26c) may now be considered as the clean quantile analog of the classical condition (4.1). Probabilistic proofs of the sufficiency of (4.1) for $F \in D(2)$ were given by Root and Rubin [30] and [2, 3]. S. Csörgő, Horváth, and Mason [5] showed that $F \in D(2)$ implies (1.27). Very recently Maller [26] proved the equivalence of $F \in D(2)$ and (1.27).

$D_p(2)$ was first characterized by Lévy [23, p. 113] by the well-known criterion

$$\liminf_{x \to \infty} x^2\{F(-x) + 1 - F(x)\} \Big/ \int_{-x}^{x} y^2 \, dF(y) = 0 \tag{4.2}$$

(cf. also [14, p. 190]). This criterion was rederived by many authors. Simons and Stout [35] and Maller [24] gave further characterizations based on F. Condition (1.29c) is the quantile analog of the classical Lévy condition (4.2). Corollary 2 in its full generality appears to be new.

The fact that stable laws have non-empty domains of attraction, and that these are the only ones that do, was proved by Lévy [21] as early as 1925 (cf. also [14, p. 162]). $D(\alpha)$ for $\alpha < 2$ was first characterized by Gnedenko [12] and Doeblin [8] independently (cf. also [14, p. 162]) by the conditions that the tail function $T(x) = F(-x) + 1 - F(x)$ is regularly varying at infinity with index $-\alpha$ and $F(-x)/T(x) \to 1 - p$ and $(1 - F(x))/T(x) \to p$ for some $p \in [0, 1]$ as $x \to \infty$. That this condition is equivalent to (1.32) in Corollary 3 was pointed out in [3], where a probabilistic proof of its sufficiency, and more generally, of the implication (1.32) $\Rightarrow$ (1.34) is given. For an earlier approach, not concerned with trimming, see also Le Page, Woodroofe, and Zinn [20].

The characterizations of $DN(2)$ and $DN(\alpha)$, $\alpha < 2$, are given on pages 181–182 of [14] in terms of F. These were first achieved for $\alpha < 2$ by Gnedenko [12]. In its full generality Corollary 4 seems to be new. (Of course, part (i) of it contains the Lindeberg–Lévy–Feller central limit theorem.)

To the best of our knowledge Corollary 5 provides the first explicit characterization of $D_p(\alpha)$, $\alpha < 2$. That the obtained necessary and sufficient condition implies the trimmed-sum statement there is also new. It is stated on page 189 of [14] that Doeblin [8] and Gnedenko [13] independently proved that $D_p(\alpha)$ is wider than $D(\alpha)$ for all $\alpha < 2$. However, the way the original sources show this is not so direct or explicit as the constructions in the proof of Corollary 5. These constructions also show that the criterion of Corollary 5 is manageable because in most cases of interest the second part of the criterion will be satisfied along the original $\{n_1\}$ and for *all* $\{r_{n_1}\}$ such that $r_{n_1} \to \infty$ and $r_{n_1}/n_1 \to 0$ as $n_1 \to \infty$.

In order to motivate further this last point consider the following example which, according to Kuti [19], is of importance in some problems of the physics of elementary particles. Define a symmetric discrete distribution by $P\{X = \pm w^k\} = p^k(1-p)/2$, $k = 0, 1, 2, \dots$, where $0 < p < 1$, $w = p^{-1/\alpha}$, $0 < \alpha < 2$. Then X has quantile function Q such that $Q(1-s) = w^k$ if $2^{-1}p^{k+1} \le s < 2^{-1}p^k$, $0 < s < \frac{1}{2}$. By simple computation

$$2^{-1/\alpha}w^{-1}s^{-1/\alpha} < n^{-1/\alpha}Q(1 - s/n) \le 2^{-1/\alpha}s^{-1/\alpha} \tag{4.3}$$

and, using (2.59), with $k(s)$ denoting the value of k for which $p^{k+1} \le s < p^k$,

$$\sigma^2(s) \sim \text{const}\, w^{2k(s)}p^{k(s)} \qquad \text{as } s \downarrow 0. \tag{4.4}$$

Now (4.4) implies that for *all* sequences $\{r_n\}$ of positive integers such that $r_n \to \infty$ and $r_n/n \to 0$, as $n \to \infty$, we have

$$\sigma(r_n/n)/\sigma(1/n) \to 0 \qquad \text{as } n \to \infty. \tag{4.5}$$

Note that by (4.3) and (4.4) there exists a subsequence $\{n'\} \subset \{n\}$ such that $\psi_j(n', s) \to \tilde{\psi}_j^{(\alpha)}(s) := -f(s)s^{-1/\alpha}$, $0 < s < \infty$, $j = 1, 2$, as $n' \to \infty$, where $2^{-1/\alpha}cw^{-1} \le f(s) \le 2^{-1/\alpha}c$ with some constant $c = c(w)$, so that by Theorem 1 the distribution of X is in the domain of partial attraction of the infinitely divisible distribution of $V_{0,0}(\tilde{\psi}_1^{(\alpha)}, \tilde{\psi}_2^{(\alpha)}, 0)$ "close" to being a stable law of exponent α. (It is interesting to note that if, keeping the symmetry, we modify the above distribution by setting $Q(1-s) = w_k$ if $2^{-1}p_{k+1} \le s < 2^{-1}p_k$, $0 < s < \frac{1}{2}$, where $p_k \downarrow 0$, $p_{k+1}/p_k \to 1$ as $k \to \infty$ and $w_k = p_k^{-1/\alpha}$, $k = 0, 1, 2, \dots$, then for the resulting F we have in fact $F \in D(\alpha)$ and not just $F \in D_p(\alpha)$.)

Concerning Corollary 6 we note that (1.35) with $W_{m,k} = N(0,1)$ and (1.36) with $W = N(0,1)$ are equivalent only by requiring (1.35) to hold for *some* $m \ge 0$ and $k \ge 0$ such that $m + k \ge 1$. This follows from the proof of the corollary. The counterpart of this result for modulus trimming was proved by Mori [28], extending an earlier result of Maller. A somewhat weaker form of the modulus trimming counterpart of the implication (1.36) $\Rightarrow$ (1.35) is given in Theorem 2 of Mori [28]. (He generally has to go down to a further subsequence of $\{n_1\}$.) The modulus trimming counterpart of the implication (1.35) for all $m \geq 1$ and $k \geq 1 \Rightarrow$ (1.36) was proved by Mori [28] in his Theorem 4.

The first characterization of D_p was given by Doeblin [8]. This was rederived by Jain and Orey [16] who also provide a new characterization in their Theorem 2.1. Unifying the approaches of this last theorem and of Maller [24], Goldie and Seneta [15] give yet another necessary and sufficient condition for $F \in D_p$. Our condition (1.38) in Corollary 7, obtained under the restrictive assumption that $F \notin D_p(2)$, is most reminiscent to the Goldie and Seneta condition. Conditions (1.40a), (1.40b), and (1.40c) in Corollary 8, given without any restriction, appear to be new as does the whole generality of this corollary concerning the trimming.

The result in Corollary 9 was first proven by Maller [24].

The first characterization of SC was stated in Theorem VIII of Doeblin [8]. His necessary and sufficient condition for $F \in SC$ was given in a polished and more transparent form in Theorem 2.5 of Jain and Orey [16]. Although it is heuristically clear in what ways this condition restricts the tail of F, it is of an inconvenient nature from the point of view of applications. Earlier Feller [11] proved that $F \in SC$ if and only if

$$\limsup_{x \to \infty} x^2\{F(-x) + 1 - F(x)\} \Big/ \int_{-x}^{x} y^2 \, dF(y) < \infty. \tag{4.6}$$

The convenient nature of this condition, as a clear relaxation of the classical domain-of-attraction conditions, has attracted a lot of attention. Here we do not cite the many important papers investigating a variety of interesting problems under the assumption of the analytic condition (4.6). Since Feller proved his result only under the assumption of symmetry and at several important steps in his proof merely sketched his arguments, we believe that the proof of Corollary 10 is the first complete proof of a characterization of SC by a condition, condition (1.42a) comparable to (4.6). To show analytically that conditions (4.1), (4.2), and (4.6) are indeed equivalent to the respective conditions (1.26c), (1.29c), and (1.42c) can be quite involved. As an example of the type of analysis that is needed to be done, in the next section we prove the equivalence of (4.6) and (1.42c). This is mandatory in view of the remarks just made.

Assuming that $F \in SC$, Pruitt [29] has shown that the arising subsequential limiting infinitely divisible laws for the whole sum $S_n(0,0)$ are not arbitrary but they all satisfy (cf. Theorem 3 for notation)

$$x^2(L(-x) + R(x)) \leq C\left\{\sigma^2 + \int_{-x}^{0} y^2 \, dL(y) + \int_{0}^{x} y^2 \, dR(y)\right\}, \qquad 0 < x < \infty, \tag{4.7}$$

where $C > 0$ is some constant and, consequently, these limiting laws have C^{∞} densities. A careful analysis similar to that used to prove the equivalence of (4.6) and (1.42c) in the next section shows that Pruitt's condition (4.7) is equivalent to our (1.44) of Corollary 10.

We note that for all the examples of F constructed in the proof of Corollary 5 and for the distribution function F of X above (4.3) we have $F \in SC$.

The equivalence of the modulus trimming counterparts of (1.41) and (1.43) was obtained by Mori [28], in his Theorem 1. Independently of us and by different methods, the equivalence of $F \in SC$ and $F \in SC(m, k)$ for all $m \geq 1$, $k \geq 1$ was proved by Maller [26].

The notion of SSC and hence Corollary 11 appear to be new.

The result in Corollary 12 is a version of a part of the theorem of de Haan and Resnick [7]. Their description of the arising subsequential limiting laws of the normalized maxima is not so full as ours.

We now briefly describe one more consequence of our approach.

Assume that $F \in D_p$ but $F \notin DN(2)$. Write $x^+ = \max(0, x)$ and $x^- = \max(0, -x)$. Set

$$S_n = -\sum_{j=1}^{n} X_j^- + \sum_{j=1}^{n} X_j^+ := -S_n^- + S_n^+.$$

By writing $Q = -Q^- + Q^+$, a straightforward extension of our methods shows that there exist a sequence of normalizing constants $A_n > 0$ and two sequences of centering constants B_n^- and B_n^+ such that along an appropriately constructed subsequence $\{n_1\} \subset \{n\}$ as $n_1 \to \infty$

$$A_{n_1}^{-1}\{S_{n_1}^- - B_{n_1}^-\} \to_D W^-,$$

$$A_{n_1}^{-1}\{S_{n_1}^+ - B_{n_1}^+\} \to_D W^+,$$

and

$$A_{n_1}^{-1}\{S_{n_1} - B_{n_1}\} \to_D W^+ - W^-,$$

where $B_{n_1} = B_{n_1}^+ - B_{n_1}^-$, W^- and W^+ are independent random variables, at least one of which is non-degenerate. For the sake of brevity the details are omitted. Results of this type were first proven by Tucker [36]. For a special case of this result when $F \in D(\alpha)$, $0 < \alpha < 2$, with a proof based on the techniques of this paper, see S. Csörgő, Horváth, and Mason [5].

Before closing this discussion we would like to emphasize that in our opinion the presented quantile function-empirical process based probabilistic approach to this classical field of probability theory frequently provides more probabilistic intuition than the classical purely analytic or Fourier analytic approach, even though it may seem unusual at first glance. It is also strong enough to produce all the listed corollaries in a unified treatment. In this respect we must underline that all the necessity statements in these corollaries come out of Theorem 5 which has a purely probabilistic proof. No probabilistic proofs were available formerly for any of these necessity statements. (See, for example, the comment of Simons and Stout [35, p. 307], concerning the lack of such a proof in case of the present Corollary 3.)

Needless to say, there are a selected few results out of the reach of the present approach which can be proved by the classical characteristic function approach. One such result is the "transitivity" theorem of Gnedenko [13] stating that if F is in the domain of partial attraction of (the infinitely divisible) law $V^{(1)}$ and $V^{(1)}$ is in the domain of partial attraction of (the infinitely divisible) law $V^{(2)}$, then F also belongs to the domain of partial attraction of $V^{(2)}$. Another, more interesting such classical result due to Doeblin [8] and Gnedenko [13] is that if $F \in D_p$ then F either belongs to the domain of partial attraction of only one type, which by Gnedenko [13] must then be a stable type, or it belongs to the domain of partial attraction of uncountably many infinitely divisible types.

Of course, a few delicate problems remain to be solved within the framework of our approach as well. Most of these are constructional problems. For example, Theorems 5, 1, 2, and 3 imply, of course, the well-known fact that all the subsequential limiting distributions of the properly normalized and centered full sum $S_n(0,0)$ are infinitely divisible. It would be interesting to prove "the incomparably deeper converse proposition" of Khinchin [18] as put by [14, p. 184–186], that all such laws indeed arise by a direct construction of the Q functions and by Theorems 1 and 2, or, for that matter, by a direct construction of Q's giving the universal laws of Doeblin [8]. Pruitt [29] gave a new proof of Khinchin's theorem in the course of showing that if $F \in SC$ then all the possible limiting laws in (4.7) indeed show up; so the latter result is also a part of this problem of constructing Q's to a given limiting triple (ψ_1, ψ_2, σ).

5. The Stochastic Compactness Condition

First we prove the implication (1.42c) $\Rightarrow$ (4.6). Consider first the case when $Q(s) < 0$ and $Q(1-s) > 0$ for all $0 < s < \delta$ for some $\delta \in (0, \frac{1}{2})$. Set

$$M(\delta) := \sup_{0<s<\delta} sG^2(s)/S^2(s) < \infty. \tag{5.1}$$

Choose $x_0 > 0$ so large that $F(-x) < \delta$ and $1 - F(x) < \delta$ for all $x > x_0$.

Notice that for all $x > x_0$,

$$Q(F(x)) \leq x \qquad \text{and} \qquad Q(F(-x)) \leq -x < 0, \tag{5.2}$$

but for all x and $\varepsilon > 0$,

$$Q(F(x) + \varepsilon) \geq x. \tag{5.3}$$

We shall use the change of variables formula

$$\int_{-x}^{x} y^2 \, dF(y) = \int_{F(-x)}^{F(x)} Q^2(u) \, du, \qquad x > 0.$$

Now by (5.1), the fact that for all $x > x_0$

$$\int_{1-F(x)-\varepsilon}^{F(x)+\varepsilon} Q^2(u) \, du \to \int_{1-F(x)}^{F(x)} Q^2(u) \, du \qquad \text{as } \varepsilon \downarrow 0$$

and (5.3) we have for all $x > x_0$,

$$\begin{aligned} M(\delta) &\geq \lim_{\varepsilon \downarrow 0} (1 - F(x) - \varepsilon) \\ &\quad \times \{Q^2(1 - F(x) - \varepsilon) + Q^2(F(x) + \varepsilon)\} \Big/ \int_{1-F(x)-\varepsilon}^{F(x)+\varepsilon} Q^2(u) \, du \\ &\geq (1 - F(x)) Q^2(F(x) +) \Big/ \int_{1-F(x)}^{F(x)} Q^2(u) \, du \\ &\geq x^2 (1 - F(x)) \Big/ \int_{1-F(x)}^{F(x)} Q^2(u) \, du. \end{aligned}$$

Also by (5.1) and (5.2), for all $x > x_0$,

$$\begin{aligned} M(\delta) &\geq F(-x) \\ &\quad \times \{Q^2(F(-x)) + Q^2(1 - F(-x))\} \Big/ \int_{F(-x)}^{1-F(-x)} Q^2(u) \, du \\ &\geq x^2 F(-x) \Big/ \int_{1-F(x)}^{F(x)} Q^2(u) \, du. \end{aligned}$$

Thus, for all $x > x_0$,

$$\begin{aligned} \max\Big\{ & x^2 F(-x) \Big/ \int_{F(-x)}^{1-F(-x)} Q^2(u) \, du, \\ & \times x^2 (1 - F(x)) \Big/ \int_{1-F(x)}^{F(x)} Q^2(u) \, du \Big\} \leq M(\delta). \end{aligned} \tag{5.4}$$

For any $x > x_0$ we have either $1 - F(x) \geq F(-x)$ or $1 - F(x) < F(-x)$. In the first situation,

$$\frac{x^2\{F(-x) + 1 - F(x)\}}{\int_{F(-x)}^{F(x)} Q^2(u)\,du} \leq \frac{2x^2(1 - F(x))}{\int_{1-F(x)}^{F(x)} Q^2(u)\,du} \leq 2M(\delta),$$

while in the second one,

$$\frac{x^2\{F(-x) + 1 - F(x)\}}{\int_{F(-x)}^{F(x)} Q^2(u)\,du} \leq \frac{2x^2F(-x)}{\int_{F(-x)}^{1-F(-x)} Q^2(u)\,du} \leq 2M(\delta)$$

by (5.4). Hence for any $x > x_0$,

$$x^2\{F(-x) + 1 - F(x)\}\Big/\int_{-x}^{x} y^2\,dF(y) \leq 2M(\delta)$$

and thus (4.6) follows.

The other two easier cases when $Q(s) \geq 0$ or $Q(s) \leq 0$ for all $s \in (0,1)$ follow by obvious modifications of the above argument. □

Finally we prove the reverse implication (4.6) $\Rightarrow$ (1.42c). This we do only in the "difficult case" when

$$Q(s) \downarrow -\infty \quad \text{and} \quad Q(1-s) \uparrow \infty \quad \text{as } s \downarrow 0. \tag{5.5}$$

The remaining cases follow in a similar fashion.

Choose $x_0 > 0$ so large that

$$x^2\{F(-x) + 1 - F(x)\}\Big/\int_{-x}^{x} y^2\,dF(y) \leq M < \infty, \qquad x > x_0, \tag{5.6}$$

for some $M > 0$. Since for any $x > x_0$ we have for all $\varepsilon > 0$ small enough that

$$(x - \varepsilon)^2\{F(-x + \varepsilon) + 1 - F(x - \varepsilon)\}\Big/\int_{-x+\varepsilon}^{x-\varepsilon} y^2\,dF(y) \leq M,$$

it follows that for all $x > x_0$,

$$x^2\{F(-x) + 1 - F(x-)\}\Big/\int_{-x}^{x-} y^2\,dF(y) \leq M. \tag{5.7}$$

Feller [11] proves that whenever (4.6) holds, then for some $C \geq 1$ and $0 < \alpha \leq 2$,

$$\limsup_{x\to\infty} \int_{-x}^{x} y^2\,dF(y)\Big/\int_{-x/\lambda}^{x/\lambda} y^2\,dF(y) \leq C\lambda^{2-\alpha} \tag{5.8}$$

for all $\lambda \geq 1$. We claim that whenever (4.6) holds, then for all $\lambda > 1$ and $s > 0$ sufficiently small depending on λ,

$$M \geq sQ^2(1-s)/\{C\lambda^{2-\alpha}S^2(s) + C\lambda^{-\alpha}sQ^2(1-s)\}. \tag{5.9}$$

Indeed, by (5.5) choose $s > 0$ sufficiently small so that $Q(s) < -x_0$ and $Q(1-s) > x_0$. Then by (5.7) and the fact that $F(Q(1-s)-) \leq 1-s$ we have

$$\begin{aligned} M &\geq Q^2(1-s) \\ &\quad \times \{F(-Q(1-s)) + 1 - F(Q(1-s)-)\} \Big/ \int_{-Q(1-s)}^{Q(1-s)-} y^2\,dF(y) \\ &\geq sQ^2(1-s) \Big/ \int_{-Q(1-s)}^{Q(1-s)-} y^2\,dF(y). \end{aligned}$$

Fix a $\lambda > 1$ and consider the two cases $Q(s) \leq -Q(1-s)/\lambda$ and $Q(s) > -Q(1-s)/\lambda$. In the first case, for all small enough $s > 0$ we obtain by (5.8)

$$\begin{aligned} \int_{-Q(1-s)}^{Q(1-s)-} y^2\,dF(y) &\leq C\lambda^{2-\alpha}\int_{-Q(1-s)/\lambda}^{Q(1-s)/\lambda} y^2\,dF(y) \\ &\leq C\lambda^{2-\alpha}\int_{Q(s)}^{Q(1-s)-} y^2\,dF(y) \\ &\leq C\lambda^{2-\alpha}\int_s^{1-s} Q^2(u)\,du. \end{aligned}$$

This implies that for all $s > 0$ small enough

$$M \geq sQ^2(1-s)/\{C\lambda^{2-\alpha}S^2(s)\},$$

which gives (5.9) in the first case. In the second case, for all small enough $s > 0$,

$$\begin{aligned} \int_{-Q(1-s)}^{Q(1-s)-} y^2\,dF(y) &\leq C\lambda^{2-\alpha}\int_{-Q(1-s)/\lambda}^{Q(1-s)/\lambda} y^2\,dF(y) \\ &\leq C\lambda^{2-\alpha}\left\{\int_{-Q(1-s)/\lambda}^{Q(s)-} y^2\,dF(y) + \int_{Q(s)}^{Q(1-s)-} y^2\,dF(y)\right\} \\ &\leq C\lambda^{-\alpha}Q^2(1-s)F(Q(s)-) + C\lambda^{2-\alpha}\int_s^{1-s} Q^2(u)\,du \\ &\leq C\lambda^{-\alpha}sQ^2(1-s) + C\lambda^{2-\alpha}S^2(s), \end{aligned}$$

which completely proves the claim in (5.9).

It is easy to see now that (5.9) implies that

$$\limsup_{s \downarrow 0} sQ^2(1-s)/S^2(s) < \infty.$$

It can be shown similarly that whenever (4.6) holds,

$$\limsup_{s \downarrow 0} sQ^2(s)/S^2(s) < \infty,$$

and these two relations together give (1.42c). □

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